Fifteenth Lecture 24-3-1947

Continued: A Special Case of Affinity

of our affinity we get. $[\Gamma^{i}_{k\ell} = \Gamma^{i}_{\ell k}]$

(4) $\frac{1}{2}(g_{\ell i,k} + g_{k\ell,i} - g_{ik,\ell}) = g_{\sigma\ell}\Gamma^{\sigma}_{ik}$

Let g^{ik} be the normalised minors of the matrix g_{ik}: $g^{ik}\, g_{i\ell} = \delta^{k}_{\ell}$

multiply (4) by $g^{\ell m}$ we get

$$\frac{1}{2} g^{\ell m}(g_{\ell i,k} + g_{k\ell,i} - g_{ik,\ell}) = g^{\ell m} g_{\sigma\ell}\Gamma^{\sigma}_{ik}$$

This is what is assumed in Einstein's 1915 theory: $\left\{ {m \atop ik} \right\} = \Gamma^{m}_{ik}$

This is the ~~Riemann~~ 3 index Christoffel curved index symbol: The square index symbol

$$= [s, ik] = \frac{1}{2}\left(\frac{\partial g_{is}}{\partial x_k} + \frac{\partial g_{ks}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_s}\right) \quad \boxed{\text{Definition}}$$

$$\left\{ {m \atop ik} \right\} = \frac{1}{2} g^{sm}\left(\frac{\partial g_{is}}{\partial x_k} + \frac{\partial g_{ks}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_s}\right)$$

That is the usual solution met with in the Einstein theory. This is a very special case of affinity.

24.3.1947:

$\Gamma^{i}_{k\ell}$: g_{ik} , ds , $d\hat{s}$

$\frac{dx_k}{d\hat{s}}$ is its own parallel transfer along the geodesic it follows that

if $\hat{s} = f(r)$ then $d\hat{s} = f'(r)\,dr$

$$\frac{dx_k}{dr} = f'(r)\frac{dx_k}{d\hat{s}}$$

$\frac{dx_k}{dr}$ is not its own parallel transfer unless $f'(r)$ is a constant. i.e

$f'(r) = a \quad \hat{s} = f(r) = ar + b$

$r = a'\hat{s} + b'$:

$$0 = -\frac{\partial\left\{ {\alpha \atop k\ell} \right\}}{\partial x_\alpha} + \frac{\partial\left\{ {\alpha \atop k\alpha} \right\}}{\partial x_\ell} + \left\{ {\alpha \atop k\beta} \right\}\left\{ {\beta \atop \alpha\ell} \right\} - \left\{ {\alpha \atop \beta\alpha} \right\}\left\{ {\beta \atop k\ell} \right\}$$

$$= R_{k\ell}$$

These are only 10 partial differential eq^{ns}

$R_{k\ell} = R_{\ell k}$ since 1st, 2nd, 3rd & 4th terms are symetric & it can be proved that $\partial\left\{ {\alpha \atop k\alpha} \right\}/\partial x_\ell = \partial\left\{ {\alpha \atop \ell\alpha} \right\}/\partial x_k$ [see the proof later]

Is there a somewhat looser connection between $\Gamma^i_{\ell k}$ & g_{ik} ? We have demanded more than is necessary.

By demanding that $g_{ik;\ell} = 0$ we have secured that $g_{ik} A^i A^k$ is its own $\parallel^{el}$ transfer for any contra~~t~~ ~~vari~~ vector A^i: We only need the $\parallel^{el}$ transfer of $\frac{dx_k}{d\hat{s}}$ i.e a vector in the direc^{n} of the tgt to the geodesic]

: Since $\Gamma^i_{k\ell}$ &

$\left\{ {i \atop k\ell} \right\}$ are two affinities

$\Gamma^{i}_{kl} - \left\{ {i \atop kl} \right\}$ is a tensor T^{i}_{kl}

where $T^{i}_{kl} = T^{i}_{lk}$

$$\Gamma^{i}_{kl} - \left\{ {i \atop kl} \right\} = T^{i}_{kl}$$

ie $\Gamma^{i}_{kl} = \left\{ {i \atop kl} \right\} + T^{i}_{kl}$: We introduce

$$g_{m\sigma} T^{\sigma}_{kl} = T_{mkl}$$

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$$g^{\rho m} g_{m\sigma} T^{\sigma}_{kl} = g^{\rho m} T_{mkl}$$

$$\therefore \delta^{\rho}_{\sigma} T^{\sigma}_{kl} = g^{\rho m} T_{mkl}$$

$$\therefore T^{\rho}_{kl} = g^{\rho m} T_{mkl}$$

$$\therefore \Gamma^{i}_{kl} = \left\{ {i \atop kl} \right\} + g^{im} T_{mkl}$$

The 2nd term causes an additional change in A^k when transferred which is

$$\delta' A^k = - \underbrace{g^{km} T_{mrs}}_{\Gamma^{k}_{rs}} A^r dx_s$$

[$dx_s = \eta A^s$ Since we are only transferring the vector A^s parallel to itself]

$$2 g_{ik} A^i \delta A^k = 0$$

$$\therefore -2 g_{ik} A^i g^{km} T_{mrs} A^r A^s \eta = 0$$

$$T_{mrs} A^m A^r A^s = 0$$

$$\therefore \sum_{(6)} T_{mrs} = 0$$

Since $T_{mrs} = T_{msr}$

$\therefore T_{mrs} + T_{rsm} + T_{smr} = 0$ or

$T_{[mrs]} = 0$

It can be shown that with these conditions we have 20 arbitrary comp's of the tensor T_{mrs}:

The same result can be reached in a quite different way which is instructive in itself:

We have seen that

$g_{ik;l} = 0$ & $\Gamma^{i}_{lk} = \Gamma^{i}_{kl}$ give uniquely

$\Gamma^{i}_{kl} = \left\{ {i \atop kl} \right\}$:

let us drop $\Gamma^{i}_{lk} = \Gamma^{i}_{kl}$:

$$g_{ik,l} - g_{\sigma k}\Gamma^{\sigma}_{il} - g_{i\sigma}\Gamma^{\sigma}_{kl} = 0 \quad \times -\tfrac{1}{2}$$

$$g_{kl,i} - g_{\sigma l}\Gamma^{\sigma}_{ki} - g_{k\sigma}\Gamma^{\sigma}_{li} = 0 \quad \times \tfrac{1}{2}$$

$$g_{li,k} - g_{\sigma i}\Gamma^{\sigma}_{lk} - g_{l\sigma}\Gamma^{\sigma}_{ik} = 0 \quad \times \tfrac{1}{2}$$

but $\Gamma^{\sigma}_{kl} = \Gamma^{\sigma}_{lk} + 2\Gamma^{\sigma}_{\underset{\vee}{kl}}$: Hence

$$g_{ik,l} - g_{\sigma k}\Gamma^{\sigma}_{il} - g_{i\sigma}\Gamma^{\sigma}_{lk} - 2g_{i\sigma}\Gamma^{\sigma}_{\underset{\vee}{kl}} = 0 \quad \times -\tfrac{1}{2}$$

$$g_{kl,i} - g_{\sigma l}\Gamma^{\sigma}_{ki} - g_{k\sigma}\Gamma^{\sigma}_{il} - 2g_{k\sigma}\Gamma^{\sigma}_{\underset{\vee}{li}} = 0 \quad \times \tfrac{1}{2}$$

$$g_{li,k} - g_{\sigma i}\Gamma^{\sigma}_{lk} - g_{l\sigma}\Gamma^{\sigma}_{ki} - 2g_{l\sigma}\Gamma^{\sigma}_{\underset{\vee}{ik}} = 0 \quad \times \tfrac{1}{2}$$