Fourteenth Lecture 3-3-1947

Applications in Relativistic Mechanics

3.3.1947:

Two pt events, diff. bet. squares of times − diff. bet. sq. of spatial coordinates = invariant

$-x_1^2 - x_2^2 - x_3^2 + x_4^2$

$-r^2 + x_4^2$

In prerelativistic mech:s both x_4 & r are constant. Here $x_4^2 - r^2$ is ~~const~~ invariant

$(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - (x_4 - y_4)^2$

is invariant where

x_1, x_2, x_3, x_4 ; y_1, y_2, y_3, y_4 are the 2 events

Take the y's $= 0$ we get

$-x_1^2 - x_2^2 - x_3^2 + x_4^2 =$ invariant. I

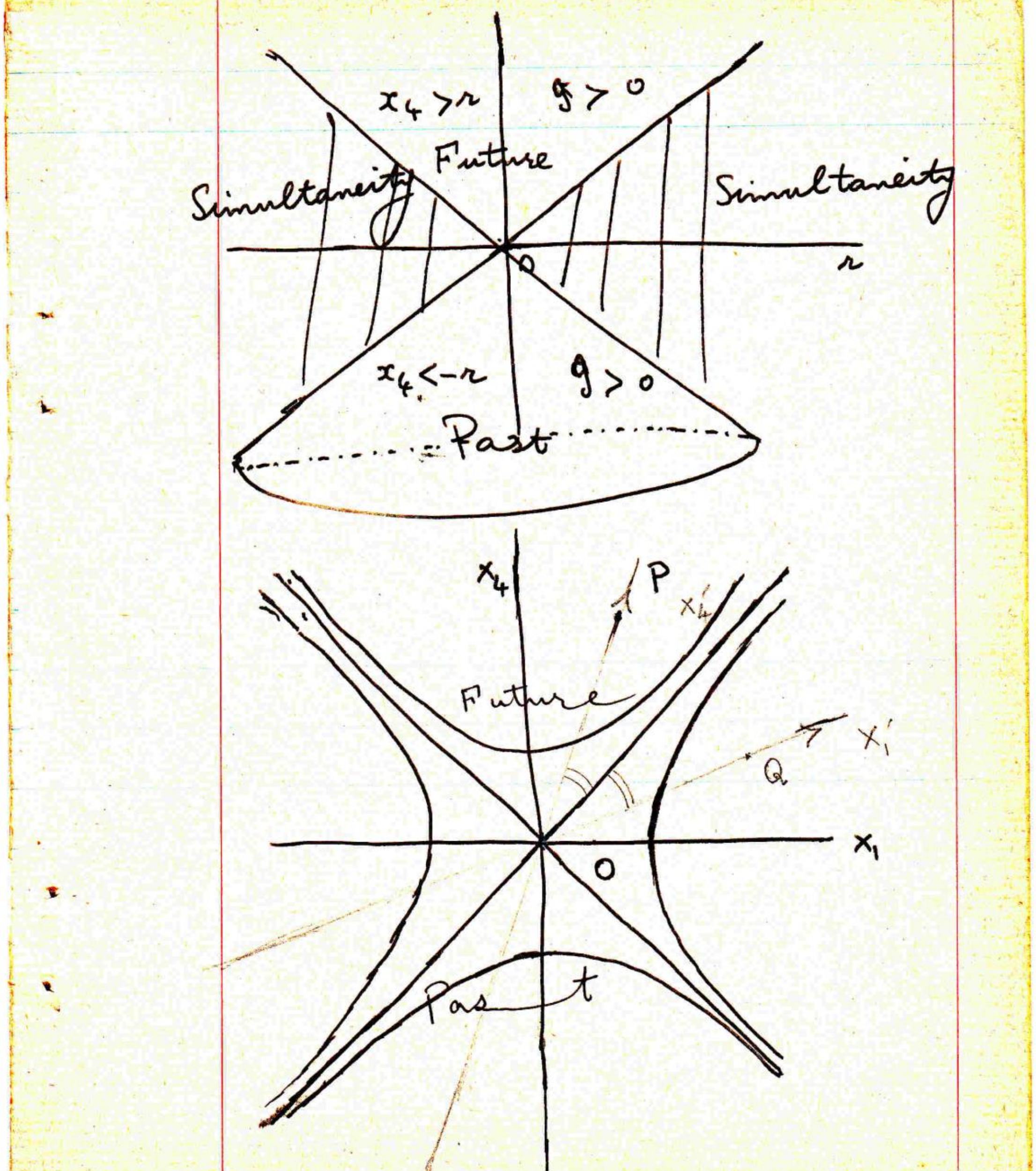
$-r^2 + x_4^2 > 0$

(i) $x_4 \neq 0$ either $x_4 > 0$ or $x_4 < 0$

By a suitable L.T. r can be made zero:

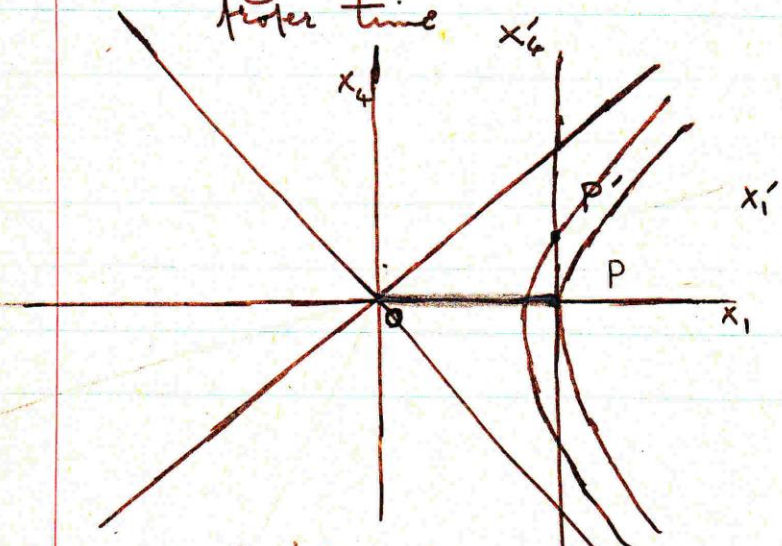
(ii) $-r^2 + x_4^2 < 0$ $r \neq 0$ in this case we can introduce a system of coordinates in which $x_4 = 0$

In the 1st Case $r = 0$ x_4 gets a certain value ~~if~~ x_4 is the minimum time interval that can elapse bet. the two events: we also have either $x_4 > |r|$ or $x_4 < -|r|$



The curves $-x_1^2 + x_4^2$ = Cons. are

hyperbolae. In this pseudo rotation these hyperbolae correspond to circles in the ordinary rotation: eigen time proper time



$|\mathcal{I}(P')| < |\mathcal{I}(P)|$

All the pts on the cone have $\mathcal{I} = 0$.
A signal of light emitted from O will reach any pt on this cone.
What are the two cones of a light disturbance emitted at any pt O?: ⟶
The 3 dimensional velocity
$\frac{dx_1}{dx_4}, \frac{dx_2}{dx_4}, \frac{dx_3}{dx_4}$ form a vector in 3 dimensions.

let

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2 > 0$$

& Consider $\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}, \frac{dx_4}{ds}$

$$ds = dx_4 \sqrt{1 - \left(\frac{dx_1}{dx_4}\right)^2 - \left(\frac{dx_2}{dx_4}\right)^2 - \left(\frac{dx_3}{dx_4}\right)^2}$$

$$ds = dx_4 \sqrt{1 - v^2}$$

$$v = (v_1, v_2, v_3) \quad \text{in 3 dimensions}$$

$\frac{v_1}{\sqrt{1-v^2}}, \frac{v_2}{\sqrt{1-v^2}}, \frac{v_3}{\sqrt{1-v^2}}, \frac{1}{\sqrt{1-v^2}}$ This is the four velocity & is a vector in 4 dimensions

How are we going to transfer these ideas to the general theory of relativity?

"A metric is an infinitesimal invariant determined by 2 neighbouring pts:"

The terminology "restricted" & "general" for the Einstein theories in 1905 & 1915 is rather misleading. Why?

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$$

$dx_k = \frac{\partial x_k}{\partial x'_l} dx'_l$ $\quad ds^2$ will transform into $ds^2 = g_{ik}\, dx_i\, dx_k$

g_{ik} is symetric → All points lying on the cone of the future represent a light signal which begins now at O & spreads. All pts on the lower cone represent a light disturbance which began at a time $-\infty$ & which reaches the point O now.

Consider 2 events $O = (0, 0, 0, 0)$

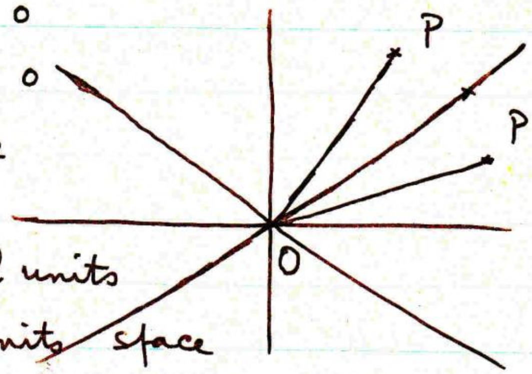
$P = (x_1, x_2, x_3, x_4)$

(i) $-r^2 + x_4^2 > 0$

(ii) $-r^2 + x_4^2 < 0$

(i) time-like

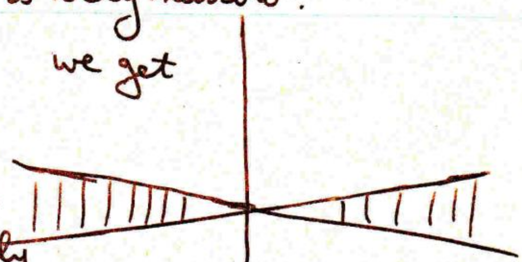
(ii) space like



We used special units

In ordinary units space of simultaneity is very narrow.

If $-r^2 + x_4^2 = 0$ we get the light cones.

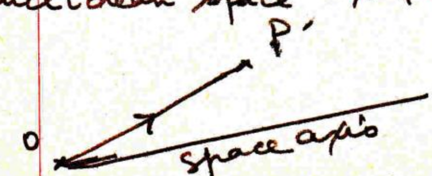


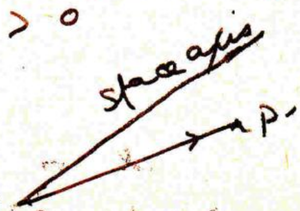
$r = \pm x_4$. These 2 lines play exactly the same role as in proper units

$y = \pm i x$ "isotropic lines"

tangents to the absolute quadric from the origin

In Euclidean space $x^2 + y^2 > 0$





O is earlier than P'   event P' is earlier than O

If O & P are on the light Cone. then a Lorentz frame can be found in which the difference in the spatial & temporal distances are as small as we like. This leads to the fact that emission & absorption of light is almost simultaneous.

Corresponding to $\overline{ds}^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$ in the special relativity we have

$$\overline{ds}^2 = g_{ik}\, dx^i\, dx^k$$

in the general relativity.

g_{ik} is a covariant vector.

Since $g'_{ik} = \frac{\partial x_m}{\partial x'_i} \frac{\partial x_n}{\partial x'_k} g_{mn}$

then $\text{Det } g'_{ik} = \left|\frac{\partial x_m}{\partial x'_j}\right|^2 \text{Det } g_{ik}$

Hence $\text{Det } g_{ik}$ does not change sign

but in $\overline{ds}^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$

$$|g_{ik}| = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1$$

$\therefore \text{Det } g_{ik} = g$ is negative.

A general transformation from x_k to x'_k contains 16 $\frac{\partial x_k}{\partial x'_j}$ but the Lorentz transf^{n} contains only 6 coefficients.

Consider $\overline{ds}^2 = -dx_1^2 - dx_2^2 - 2dx_3\,dx_4$

$= -dy_1^2 - dy_2^2 - dy_3^2 - dy_4^2$

$dy_1 = dx_1 \quad dy_2 = dx_2$

$dy_3 = \frac{1}{\sqrt{2}}(dx_3 + dx_4) \qquad dy_4 = \frac{1}{\sqrt{2}}(dx_3 - dx_4)$

The determinant of g_{ik} is not zero.

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

We can get an element ds with 2 time-like Coordinates & 2 space like Coordinates

$\overline{ds}^2 = -\alpha dx_1^2 - \alpha dx_2^2 + 2dx_3\,dx_4 + \epsilon dx_3^2 + \epsilon dx_4^2$

$$g = \begin{vmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & \epsilon & -1 \\ 0 & 0 & -1 & \epsilon \end{vmatrix} = (-1+\epsilon^2)\,\alpha^2$$

$\alpha = \frac{1}{\sqrt{1-\epsilon^2}}$ if $g = -1$

The general $\overline{ds}^2$ need not be interpretable

We can have ~~the~~ 4 time like elements

How can We Adopt the Metrical Connection and the Affine Connection?

How can we adapt the metrical connection & the affine connection ?

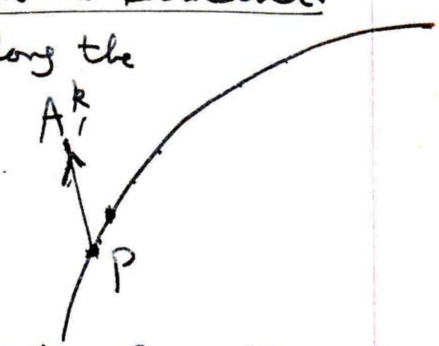
g_{ik} shd not contradict the rudementary metric given by the affine connection & defined by the element of natural length along the geodesics defined by the affine connection :

Γ^{i}_{kl} : g_{kl} : $ds^2 = g_{ik}\, dx^i dx^k$

Let $\hat{s}$ be arc along the geodesic $d\hat{s}$ is the element of length along " " : The invar^t $g_{ik} A^i A^k$ "does not change on the "parallel transfer of the vector A along any geodesic". should remain constant. A^i being any arbitrary contr^t vector :: : This is a Demand:

$g_{ik}\frac{dx_i}{d\hat{s}}\frac{dx_k}{d\hat{s}}$ is a cons. along the geodesic by taking

$$A^k = \frac{dx^k}{d\hat{s}}$$



$$\therefore g_{ik}\frac{dx_i}{d\hat{s}}\cdot\frac{dx_k}{d\hat{s}} = C$$

$\sqrt{C}\; d\hat{s} = d\hat{\hat{s}}$ we get $g_{ik}\frac{dx_i}{d\hat{\hat{s}}}\frac{dx_k}{d\hat{s}}$

$$g_{ik}\frac{dx_i}{d\hat{\hat{s}}}\frac{dx_k}{d\hat{\hat{s}}} = 1$$

$d\hat{\hat{s}}$ is the line element of the field metric g_{ik} parallel transfered

The demand $g_{ik} A^i A^k$ is little more than

$$g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} = 1$$

The demand is sufficient but not necessary

The demand means that g_{ik} is its own parallel transfer. i.e the invariant derivative of g_{ik} is zero w. r. to our affinity: according to our demand

Proof: $g_{ik} A^i A^k$ remains unchanged when A^i & A^k are $\parallel^{el}$ transferred & g_{ik} the field takes its value at the new point on the geodesic. but if we find the parallel transfer of the three factors we get the $\parallel^{el}$ transfer of the invariant which is the same: Hence the result stated & underlined above:

$$-\tfrac{1}{2} \quad g_{ik,l} - g_{\sigma k} \Gamma^{\sigma}_{il} - g_{i\sigma} \Gamma^{\sigma}_{kl} = 0 \quad (1)$$

We make a cyclical permutation $\boxed{\begin{matrix} i & k & l \\ l & k & i \\ l & i & k \end{matrix}}$

$$+\tfrac{1}{2} \quad g_{kl,i} - g_{\sigma l} \Gamma^{\sigma}_{ki} - g_{k\sigma} \Gamma^{\sigma}_{li} = 0 \quad (2)$$

$$+\tfrac{1}{2} \quad g_{li,k} - g_{\sigma i} \Gamma^{\sigma}_{lk} - g_{l\sigma} \Gamma^{\sigma}_{ik} = 0 \quad (3)$$

Adding & making use of the symetry