Eleventh Lecture 27-1-1947

Curvature Tensor in B^i_{klm}

27.1.1947

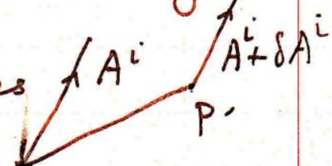
Curvature Tensor is B^i_{klm}

The contraction $B^i_{kli} = R_{kl}$ this is the fundamental tensor in the whole theory.

Geodesics in Affine Connection: Book by "Eisenhart ~~Non~~-Riemanian Geometry".

$$\delta A^i = -\Gamma^i_{kl} A^k dx_l$$

The notion of a Geodesic arises from finding at P' the parallel transfer of the vector PP' at P.



then finding at P'' the parallel transfer of the vector P'P'' & so on. In the limit we obtain a geodesic.

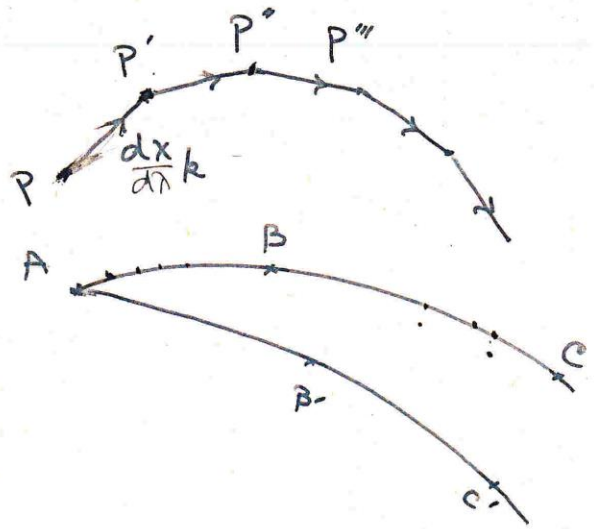
The affine connection allows us to make a comparison of lengths which is a metrical property:

$$\frac{\text{Length BC}}{\text{Length AB of the same geodesic}} = \lim \frac{\text{Number of line elements in BC}}{\text{" " " " in AB}}$$

: but we can't compare the lengths BC, B'C' of two geodesics. The affine connection does not afford

Differential Equation of a Geodesic:

The best way is to find the 4 coordinates as functions of one parameter.



any means for this comparison.

Let

$x_1(\lambda)$, $x_2(\lambda)$, & $x_3(\lambda)$, $x_4(\lambda)$.

express the coordinates of any pt on our geodesic in terms of a parameter λ:

$\frac{dx_k}{d\lambda}$ indicates the direction of the curve. This vector when parallel transferred to the neighbouring pt gives the direction of the curve at the neighbouring pt. Hence we have

$$\frac{dx_k}{d\lambda} - \Gamma^k{}_{lm} \frac{dx_l}{d\lambda} dx_m = M\left(\frac{dx_k}{d\lambda} + \frac{d^2x_k}{d\lambda^2} d\lambda\right)$$

$$\therefore M \frac{d^2x_k}{d\lambda^2} + \Gamma^k{}_{lm} \frac{dx_l}{d\lambda}\frac{dx_m}{d\lambda} = \frac{1-M}{d\lambda}\,\frac{dx_k}{d\lambda} \quad (1)$$

in the limit M must tend to 1

put $1 - M = \phi(\lambda)\, d\lambda$ Hence

the differential eq^n of the geodesic is

$$M \frac{d^2x_k}{d\lambda^2} + \Gamma^k{}_{lm} \frac{dx_l}{d\lambda} \cdot \frac{dx_m}{d\lambda} = \phi(\lambda) \frac{dx_k}{d\lambda}$$

Will our $d\lambda$ be a natural measure of length for our geodesic? How this eq^n changes when we change λ? Let us change λ into an arbitrary monotonic function $s(\lambda)$

$$\frac{dx_k}{d\lambda} = \frac{dx_k}{ds}\,\frac{ds}{d\lambda} = \frac{dx_k}{ds}\, s'$$

$$\frac{d^2x_k}{d\lambda^2} = \frac{dx_k}{ds} \cdot \underbrace{\frac{d^2s}{d\lambda^2}}_{s''} + \frac{d^2x_k}{ds^2} \underbrace{\left(\frac{ds}{d\lambda}\right)^2}_{s'^2}$$

$$\frac{d^2x_k}{ds^2}s'^2+\frac{dx_k}{ds}s''+\Gamma^k{}_{lm}\frac{dx_k}{ds}\cdot\frac{dx_m}{ds}s'^2=\phi\frac{dx_k}{ds}s'$$

$$\therefore \frac{d^2x_k}{ds^2}+\Gamma^k{}_{lm}\frac{dx_k}{ds}\cdot\frac{dx_m}{ds}=\frac{\phi s'-s''}{s'^2}\,\frac{dx_k}{ds}$$

If $S(\lambda)$ is chosen in such a way that the R.H.S vanishes. $\frac{\phi s'-s''}{s'^2}=0$

$$\phi s'-s''=0 \qquad \frac{s''}{s'}=\phi=\frac{d(\log s')}{d\lambda}$$

$$\therefore \log s'=\int^{\lambda}\phi\,d\lambda \qquad s'=e^{\int^{\lambda}\phi\,d\lambda}$$

$$\therefore S(\lambda)=\int^{\lambda}e^{\int^{\lambda'}\phi\,d\lambda'}\cdot d\lambda'$$

$$\frac{d^2x_k}{ds^2}+\Gamma^k{}_{lm}\frac{dx_k}{ds}\cdot\frac{dx_m}{ds}=0$$

This s apart from $[\hat{S}=a+bs]$ is uniquely determined & is the natural measure of length along the geodesic. This could be obtained by putting the R.H.S of (1) equal to zero

$$M\frac{d^2x_k}{d\lambda^2}+\Gamma^k{}_{lm}\frac{dx_k}{d\lambda}\cdot\frac{dx_m}{d\lambda}=\phi(\lambda)\frac{dx_k}{d\lambda}$$

$$\Gamma^k{}_{\underline{lm}}+\underbrace{\Gamma^k{}_{\underset{\vee}{lm}}}_{=0}$$

$$\Gamma^k{}_{lm}$$

a Skew affinity may be fundamental in the description of nature:

Is there any other way of modifying a given affinity so as to leave its geodesics unchanged?: Given $\Gamma^k{}_{lm}$

Let V_l be a covariant vector field. $\hat{\Gamma}^k{}_{lm} = \Gamma^k{}_{lm} + \delta^k{}_l V_m + \delta^k{}_m V_l$ does not change the geodesics.

Proof: Choosing our parameter s the natural length of measure

$$\frac{d^2X_k}{ds^2} + \Gamma^k{}_{lm}\frac{dX_l}{ds}\cdot\frac{dX_m}{ds} = 0$$

$$\frac{d^2X_k}{ds^2} + \Gamma^k{}_{lm}\frac{dX_k}{ds}\frac{dX_l}{ds} + V_m\frac{dX_k}{ds}\cdot\frac{dX_m}{ds} + V_l\frac{dX_k}{ds}$$

$$+ 2V_m\frac{dX_k}{ds}\frac{dX_m}{ds} = 0$$

$$= -2V_m\frac{dX_k}{ds}\frac{dX_m}{ds}$$

$$\frac{d}{ds}\left[\int V_m\, dX_m\right]$$

This does not change the geodesics but it does change the natural measure of length. It can be proved that the only way to modify our affinity so as to leave the geodesic & natural measure of length unchanged is to take a general affinity since —

In a geodesic the skew part has no influence .i.e. The geodesic is not influenced at all by the skew part of our affinity

$$\frac{dx^l}{ds} = 0$$