Application of the Single Particle Schrödinger Fluid to the p-Shell Nuclei

S. B. Doma

Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria, Egypt.

E-mail: sbdoma@alexu.edu.eg

Abstract

The concept of the single-particle Schrödinger fluid is adopted for some nuclei in the p-shell, namely: ^{6}Li, ^{7}Li, ^{8}Li, ^{9}Li, ^{10}Li, ^{11}Li, ^{8}Be, ^{9}Be, ^{10}B, ^{11}B, and ^{12}C, to obtain the cranking-model, the rigid body-model and the equilibrium-model moments of inertia of these nuclei. In terms of these moments of inertia we developed two new simple formulas to calculate the electric quadrupole moments from the cranked and the rigid-body moments of inertia. Accordingly, the quadrupole moments of these deformed nuclei are obtained as functions of their moments of inertia and the deformation parameter β. To represent the interaction of one nucleon with the nuclear field, we take the one introduced by Nilsson for the axially deformed nuclei. Variations of the moments of inertia and the electric quadrupole moments of these nuclei with respect to the deformation parameters of these nuclei are also given. The obtained numerical results in the case of the cranking-model moments of inertia for all the considered nuclei and the rigid body-model moments of inertia, for some nuclei, are in good agreement with the corresponding experimental values, which showed that the adopted model for these nuclei is adequate.

Keywords: Deformed nuclei, p-shell nuclei, single-particle Schrödinger fluid, moment of inertia, electric quadrupole moment.

1. Introduction

Nuclear Structure, theory and applications, is a very attractive topic of research in physics and engineering, due to its wide range of subjects and close connections with other topics, such as: quantum mechanics, group theory, fluid mechanics, algebra, computer science, statistical physics, field theory, nuclear industry, and many other subjects [1-23].

The quantum fluid [24-47] is completely transparent internally with respect to motion of the constituent particles, and to receive disturbances solely by way of surface deformations. Its near incompressibility comes about, not by particle-to-particle push, as in an ordinary liquid, but by more subtle means. It is capable of collective oscillations, but it is the wall which organizes these disturbances, not nucleon-to-nucleon interactions. Oscillations experience a damping, but the mechanism of the damping is unlike that encountered in ordinary liquids. The rotational properties of the quantum fluid are quite different from those of ordinary fluids.

The absolute values of the rotational energies or equivalently the moments of inertia require knowledge of the fine details of the intrinsic nuclear structure. Consequently, the investigation of the nuclear moments of inertia is a sensitive check for the validity of the nuclear structure theories [48,49].

Moreover, the study of the velocity fields for the rotational motion led to the formulation of the so-called the single-particle Schrödinger fluid [50-59]. Since the Schrödinger-fluid theory is at present an independent particle model, the cranking model approximation for the velocity fields and the moments of inertia play the dominant role in this theory.

The constituent particle of the nucleus, nucleon, in the single-particle Schrödinger fluid, is assumed to move independently in an average potential, assumed deformed, which represents the action of all the other particles on this nucleon. The governing equation in this concept is the single-particle time-dependent Schrödinger wave equation, which is of a parabolic type of second-order partial differential equation.

The solutions of this equation give the characteristics properties of the wave-type-character of the nucleon, which is moving in the average field created due to the presence of the other $(A-1)$ particles, where A is the number of nucleons in the nucleus. The single particle Schrödinger fluid is a concept which is used to describe the motion of a single nucleon in an axially deformed potential of the nucleus. It is not simply a fluid motion as in fluid mechanics but a probability current density associated with the solution of the considered wave equation. This concept is carried out by a suitable choice of the time- dependent part of the nucleon wave function in the time-dependent Schrödinger wave equation. This concept can be applied to study the rotational motion of a deformed nucleus [60-93].

In the present paper, we carried out the derivations of the concept of the single-particle- Schrödinger fluid and accordingly clarified how the cranked-model and the rigid body-model moments of inertia of an axially deformed nucleus can be obtained in framework of this concept and by applying also the basics of the Nilsson model [66]. As examples for the applications of this concept to the calculations of the nuclear moments of inertia, we have calculated the cranking-model and the rigid body-model moments of inertia of some nuclei in the p-shell nuclei, namely: the nuclei ^{6}Li, ^{7}Li, ^{8}Li, ^{9}Li, ^{10}Li, ^{11}Li, ^{8}Be, ^{9}Be, ^{10}B, ^{11}B, and ^{12}C. The variations of the nuclear moments of inertia with respect to the deformation parameter β have been also given in this paper. Furthermore, we derived a new relation depending on the nuclear moments of inertia for a given nucleus and the electric quadrupole moment of this nucleus. Accordingly, we applied this formula to calculate the electric quadrupole moments of the eight nuclei: ^{6}Li, ^{7}Li, ^{8}Li, ^{8}Be, ^{9}Be, ^{10}B, ^{11}B, and ^{12}C.

2. The Fluid Dynamical Equations

We assume that each nucleon (proton or neutron) with mass M, in a nucleus consisting of A nucleons, is moving independently in a single-particle potential $V\big(\boldsymbol{r}, a(t)\big)$, which is deformed with time t, through its parametric dependence on a classical shape variable $\alpha(t)$. Here, $\alpha(t)$ is assumed to be an externally prescribed function of t. Thus, the Hamiltonian for the present problem is given by [50,51]

$$H\big(\boldsymbol{r}, \boldsymbol{v}, a(t)\big) = -\frac{\hbar^2}{2M}\nabla^2 + V\big(\boldsymbol{r}, a(t)\big). \tag{1}$$

The single-particle time-dependent wave function $\Psi(\mathbf{r}, \alpha(t), t)$ which is a well-behaved function and satisfies the time-dependent Schrödinger wave equation, that describes the motion of a nucleon, is defined as:

$$\mathrm{H}\big(\boldsymbol{r}, \boldsymbol{v}, \alpha(t)\big)\Psi(\boldsymbol{r}, \alpha(t), t) = i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r}, \alpha(t), t). \tag{2}$$

To obtain a fluid dynamical description of the wave function $\Psi(\boldsymbol{r}, \alpha(t), t)$, we used the polar form of the wave function. We first isolated the explicit time dependence as follows [50]:

$$\Psi(\boldsymbol{r}, \alpha(t), t) = \psi\big(\boldsymbol{r}, \alpha(t)\big)\exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon\big(\alpha(t')\big)dt'\right\}, \tag{3}$$

where ϵ is the energy density which depends on the time through the parameter $\alpha(t)$. Then, we write the complex wave function $\psi(\boldsymbol{r}, \alpha(t))$ in the following polar form [50]:

$$\psi\big(\boldsymbol{r}, \alpha(t)\big) = \Phi\big(\boldsymbol{r}, \alpha(t)\big)\exp\left\{-\frac{iM}{\hbar}S\big(\boldsymbol{r}, \alpha(t)\big)\right\}, \tag{4}$$

where $\Phi(\boldsymbol{r}, \alpha(t))$ and $S(\boldsymbol{r}, \alpha(t))$ are assumed to be real functions of $\boldsymbol{r}$ and $\alpha(t)$. Finally, we assume that the function $\Phi(\boldsymbol{r}, \alpha(t))$ is positive definite. In the case of rotation, the parameter $\alpha(t)$ becomes the angle of rotation, $\theta = \Omega t$, where Ω is the angular velocity.

Substituting equations (1), (3) and (4) into (2) we get:

$$H\Psi(\boldsymbol{r}, \alpha(t), t) = \exp\left\{-\frac{i}{\hbar}\int_0^t \varepsilon\big(\alpha(t')\big)dt'\right\}\left\{-\frac{\hbar^2}{2M}\nabla^2 + V\big(\boldsymbol{r}, \alpha(t)\big)\right\} \times \left\{\Phi\big(\boldsymbol{r}, \alpha(t)\big)\exp\left\{-\frac{iM}{\hbar}S\big(\boldsymbol{r}, \alpha(t)\big)\right\}\right\} = i\hbar\frac{\partial}{\partial t}\left[\Phi\big(\boldsymbol{r}, \alpha(t)\big)\exp\left\{-\frac{iM}{\hbar}S\big(\boldsymbol{r}, \alpha(t)\big)\right\} \times \exp\left\{-\frac{i}{\hbar}\int_0^t \varepsilon\big(\alpha(t')\big)dt'\right\}\right] \tag{5}$$

So that:

$$H\Psi(\boldsymbol{r}, \alpha(t), t) = \exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon\big(\alpha(t')\big)dt'\right\}\exp\left\{-\frac{iM}{\hbar}S(\boldsymbol{r}, \alpha(t)\right\} \times \left[\epsilon(\alpha(t)\Phi(\boldsymbol{r}, \alpha(t) + M\Phi\big(\boldsymbol{r}, \alpha(t)\big)\frac{\partial}{\partial t}S\big(\boldsymbol{r}, \alpha(t)\big) + i\hbar\frac{\partial}{\partial t}\Phi\big(\boldsymbol{r}, \alpha(t)\big)\right]. \tag{6}$$

Hence,

$$\left\{-\frac{\hbar^2}{2M}\nabla^2+V\big(\boldsymbol{r},\alpha(t)\big)\right\}\left\{\Phi\big(\boldsymbol{r},\alpha(t)\big)\exp\left\{-\frac{iM}{\hbar}S\big(\boldsymbol{r},\alpha(t)\big)\right\}\right\}=$$
$$\exp\left\{-\frac{iM}{\hbar}S\big(\boldsymbol{r},\alpha(t)\big)\right\}\Big[\epsilon\big(\alpha(t)\big)\Phi\big(\boldsymbol{r},\alpha(t)\big)+M\Phi\big(\boldsymbol{r},\alpha(t)\big)\frac{\partial}{\partial t}S(\boldsymbol{r},\alpha(t)+$$
$$i\hbar\frac{\partial}{\partial t}\Phi\big(\boldsymbol{r},\alpha(t)\big)\Big]. \tag{7}$$

But we know that;

$$\nabla^2\left(\Phi\exp\left\{-i\frac{MS}{\hbar}\right\}\right)=(\nabla^2\Phi)\exp\left\{-i\frac{MS}{\hbar}\right\}+\Phi\nabla^2\left(\exp\left\{-i\frac{MS}{\hbar}\right\}\right)$$
$$+2(\nabla\Phi)\cdot\nabla\left(\exp\left\{-i\frac{MS}{\hbar}\right\}\right), \tag{8}$$

Also, we have

$$\nabla\left(\exp\left\{-i\frac{MS}{\hbar}\right\}\right)=-i\frac{M}{\hbar}(\nabla S)\exp\left\{-i\frac{MS}{\hbar}\right\},$$

and

$$\nabla^2\left(\exp\left\{-i\frac{MS}{\hbar}\right\}\right)=-i\frac{M}{\hbar}\nabla\cdot\left[(\nabla S)\exp\left\{-i\frac{MS}{\hbar}\right\}\right]$$
$$=-i\frac{M}{\hbar}\left[(\nabla^2 S)\exp\left\{-i\frac{MS}{\hbar}\right\}-i\frac{M}{\hbar}(\nabla S)\cdot(\nabla)\exp\left\{-i\frac{MS}{\hbar}\right\}\right]$$
$$=-i\frac{M}{\hbar}(\nabla^2 S)\exp\left\{-i\frac{MS}{\hbar}\right\}-\frac{M^2}{\hbar^2}(\nabla S)\cdot(\nabla S)\exp\left\{-i\frac{MS}{\hbar}\right\}.$$

Substituting from the above results into equation (6) we get:

$$i\hbar\frac{\partial\Phi}{\partial t}\exp\left\{-i\frac{MS}{\hbar}\right\}+M\Phi\frac{\partial S}{\partial t}\exp\left\{-i\frac{MS}{\hbar}\right\}+\epsilon\Phi\exp\left\{-i\frac{MS}{\hbar}\right\}=$$
$$-\frac{\hbar^2}{2M}\exp\left\{-i\frac{MS}{\hbar}\right\}\nabla^2\Phi+\frac{M}{2}\Phi\exp\left\{-i\frac{MS}{\hbar}\right\}(\nabla S)\cdot(\nabla S)+\frac{i\hbar}{2}\Phi\exp\left\{-i\frac{MS}{\hbar}\right\}(\nabla^2 S)+$$
$$i\hbar(\nabla\Phi)\cdot(\nabla S)\exp\left\{-i\frac{MS}{\hbar}\right\}+V\Phi\exp\left\{-i\frac{MS}{\hbar}\right\}. \tag{9}$$

Dividing all the terms by $\exp\left\{-i\frac{MS}{\hbar}\right\}$ we get:

$$i\hbar\frac{\partial\Phi}{\partial t}+M\Phi\frac{\partial S}{\partial t}+\epsilon\Phi=-\frac{\hbar^2}{2M}\nabla^2\Phi+\frac{i\hbar}{2}\Phi(\nabla^2 S)+\frac{M}{2}\Phi(\nabla S)^2+i\hbar(\nabla\Phi)\cdot(\nabla S)+V\Phi.$$

Hence,

$$i\hbar\left[\frac{\partial\Phi}{\partial t}-\frac{1}{2}\Phi(\nabla^2 S)-(\nabla\Phi)\cdot(\nabla S)\right]=\left\{-M\Phi\frac{\partial S}{\partial t}-\epsilon\Phi+H\Phi+\frac{M}{2}\Phi(\nabla S)\cdot(\nabla S)\right\}. \tag{10'}$$

The only accepted solutions of equation (10) can be obtained by equating the real and imaginary parts to zero. We then obtain a coupled equations for Φ and S as follows:

$$\left[H-M\left(\frac{\partial S}{\partial t}-\frac{1}{2}\nabla S\cdot\nabla S\right)\right]\Phi=\epsilon\Phi. \tag{11}$$

and

$$\frac{1}{2}\Phi(\nabla^2 S) + (\nabla\Phi)\cdot(\nabla S) = \frac{\partial\Phi}{\partial t}. \tag{12}$$

We may call equation (11) modified time-independent Schrödinger equation because it differs from the usual time-independent Schrödinger equation $H\Phi = \epsilon\Phi$ by an added term which we refer to as the dynamical modification potential:

$$V_{dyn} = -M\left[\frac{\partial S}{\partial t} - \frac{1}{2}(\nabla S)\cdot(\nabla S)\right]. \tag{13}$$

When we identify the probability density of the single particle as the square of the amplitude $|\Phi|^2$ and recognize that equation (12), when multiplied by 2Φ, gives

$$\Phi^2\nabla^2 S + \nabla\Phi^2\cdot\nabla S = \frac{\partial\Phi^2}{\partial t}, \tag{14}$$

then, we will obtain two equations, the first is:

$$\rho\nabla\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\rho = -\frac{\partial\rho}{\partial t}, \tag{15}$$

where $\boldsymbol{v}$ is the irrotational velocity and ρ is the density. It is the well-known equation of continuity in fluid mechanics. It can be rewritten in the form:

$$\nabla\cdot(\rho\boldsymbol{v}) = -\frac{\partial\rho}{\partial t}, \tag{16}$$

where $\rho = \Phi^2$ and $\boldsymbol{v} = -\nabla S$.

The second equation is

$$(H + V_{dyn})\Phi = \epsilon\Phi, \tag{17}$$

which is a modified Schrödinger equation with:

$$V_{dyn} = -M\left(\frac{\partial S}{\partial t} - \frac{1}{2}\boldsymbol{v}^2\right). \tag{18}$$

From equation (4) we can easily obtain:

$$S = \frac{i\hbar}{2M}\ln\left(\frac{\psi}{\psi^*}\right). \tag{19}$$

We know that,

$$\psi = \Phi\exp\left\{-i\frac{MS}{\hbar}\right\}, \qquad \psi^* = \Phi\exp\left\{i\frac{MS}{\hbar}\right\},$$

and

$$\boldsymbol{v} = -\nabla S = -\frac{i\hbar}{2M}\nabla\left(\ln\left(\frac{\psi}{\psi^*}\right)\right) = -\frac{i\hbar}{2M}[\nabla(\ln(\psi)) - \nabla(\ln(\psi^*))] = \frac{i\hbar}{2M}\left[\frac{\nabla\psi^*}{\psi^*} - \frac{\nabla\psi}{\psi}\right].$$

Therefore,

$$\boldsymbol{v} = \frac{i\hbar}{2M|\psi|^2}[\psi\nabla\psi^* - \psi^*\nabla\psi].$$

The current of the single particle state is defined by

$$\boldsymbol{j} = \rho\boldsymbol{v} = \frac{i\hbar\rho}{2M|\psi|^2}[\psi\nabla\psi^* - \psi^*\nabla\psi].$$

Putting $\rho = |\Phi|^2$, we get

$$\boldsymbol{j} = \frac{i\hbar}{2M}\frac{|\Phi|^2}{|\psi|^2}[\psi\nabla\psi^* - \psi^*\nabla\psi].$$

Since,

$$|\psi|^2 = |\Phi|^2\left|e^{-\frac{iMS}{\hbar}}\right|^2 = |\Phi|^2,$$

we finally obtain:

$$\boldsymbol{j} = \frac{i\hbar}{2M}[\psi\nabla\psi^* - \psi^*\nabla\psi]. \tag{20}$$

Euler's equation for the non-viscous fluid flow is given by:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho}, \tag{21}$$

where p is the pressure on the fluid at a point $P(\boldsymbol{r})$ at an instant of time t. For an ideal fluid, ∇p is related to the enthalpy per unit mass, w, of the fluid by the following manner

$$\frac{\nabla p}{\rho} = \nabla w. \tag{22}$$

Therefore, Euler's equation can be rewritten as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla w. \tag{23}$$

After integration and using $\boldsymbol{v} = -\nabla s$ we get

$$\frac{\partial s}{\partial t} - \frac{1}{2}v^2 = w. \tag{24}$$

Using also

$$v^2 = (\nabla S)^2,$$

we get

$$\frac{\partial S}{\partial t} - \frac{1}{2}(\nabla S)^2 = w, \tag{25}$$

where S is the velocity potential for $\boldsymbol{v}$, ($\boldsymbol{v} = -\nabla S$) and the constant of integration in equation (24) is chosen here to be zero. Therefore, we can write:

$$V_{dyn} = -M\left[\frac{\partial S}{\partial t} - \frac{1}{2}(\nabla S)^2\right], \tag{26}$$

$$V_{dyn} = -Mw, \tag{27}$$

and then the modified Schrödinger equation takes the form

$$(H - Mw)\Phi = \epsilon\Phi, \tag{28}$$

where w is now the enthalpy of the single-particle Schrödinger fluid.

Hence, we have a set of fluid dynamical equations completely analogous to those which describe a classical fluid. This set consists of the continuity equation (15), the Euler equation (25), and an equation of state (28). By derivation, their content is precisely that of the original time-dependent Schrödinger equation. Hill and Wheeler [41] assumed that the single-practice Schrödinger fluid is irrotational and implicitly incompressible flow. The present formulation is specifically not restricted to incompressible flows but allows also irrotational but compressible.

The description of the density $|\psi|^2$ as a classical fluid implies that we are assigning labels to each mass element $|\psi|^2\Delta x\Delta y\Delta z$, and considering its motion in time, as described by the velocity field $\boldsymbol{v}$. However, in quantum mechanics, the quantity $|\psi|^2\Delta x\Delta y\Delta z$ is interpreted as the probability of finding the nucleon in the volume element $\Delta x\Delta y\Delta z$.

In addition to the irrotational velocity $\boldsymbol{v}$, which is a result from the fluid dynamical equation, other velocity fields which satisfy the continuity equation of the Schrödinger equation occur. Among these velocity fields are [50] the incompressible velocity field, the regular velocity field, the geometric velocity field, and the rigid body velocity field. For rotations, the rigid-body velocity field v_{rig} is defined as:

$$\boldsymbol{v}_{rig} = \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{29}$$

It is seen that this velocity field is incompressible, regular and of geometric type.

In the adiabatic approximation, where $\frac{\partial\alpha}{\partial t} \to 0$, that is the angle of rotation θ is constant of the time, the collective kinetic energy of a nucleon in the nucleus is given by [50]

$$T_K = \frac{1}{2}\int \rho\boldsymbol{v}_K \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})dr. \tag{30}$$

and the collective kinetic energy T of the nucleus is given by

$$T = \frac{1}{2}\mathrm{M}\int \rho_T\boldsymbol{v}_T \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})dr, \tag{31}$$

where ρ_T is the total density distribution of the nucleus and $\boldsymbol{v}_T$ is the total velocity field,

$$\boldsymbol{v}_T = \frac{\sum_{occ} \rho_K \boldsymbol{v}_K}{\sum_{occ} \rho_K}. \tag{32}$$

The collective kinetic energy for the entire nucleus, as an integral of a density weighted quadratic form in velocities, conforms to the classical structure of a continuum kinetic energy, but involves two distinct velocity fields, rather than simply the square of a single velocity field. The occurrence of these two distinct velocity fields reflects the two essential aspects of the cranking motion:

(i) The rotation of the potential well, which can (just as can the density's variation in time for pure rotation) be described as the regular velocity field $\boldsymbol{\Omega} \times \boldsymbol{r}$.

(ii) The response of the individual particle to the motion of the potential, can be described by $\boldsymbol{v}_K$. This structure seems eminently natural, and perhaps sufficiently general to encompass a wide variety of the kinetic energy forms.

3. The nucleon-nucleon potential

To represent the interaction of one nucleon with the nuclear field, created by the other nucleons, we take the one introduced by Nilsson for the axially deformed nuclei [66]:

$$V(\boldsymbol{r}, \alpha(t)) = \frac{1}{2} M\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right) + C\boldsymbol{\ell}.\boldsymbol{s} + D\ell^2 \tag{33}$$

where:

$$\omega_z^2 = \omega_0^2 \left(1 - \frac{4}{3}\delta\right), \tag{34}$$

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(1 + \frac{2}{3}\delta\right). \tag{35}$$

The condition of constant volume of the nucleus leads to

$$\omega_x \omega_y \omega_z = \text{const.} \tag{36}$$

Keeping this condition in the general case together with (34) and (35), ω_0 has to depend on δ in the following way:

$$\omega_0(\delta) = \omega_0^0 \left\{1 - \frac{12}{9}\delta^2 - \frac{16}{27}\delta^3\right\}^{-\frac{1}{6}}, \tag{37}$$

where ω_0^0 is the value of $\omega_0(\delta)$ for $\delta = 0$. The parameter $\hbar\omega_0^0$ is known as the non-deformed oscillator parameter. This parameter can be calculated from the values of the total number of protons in the nucleus *Z*, the number of neutrons *N* and the mass number *A* as follows [80]:

$$\hbar\omega_0^0 = \frac{38.6A^{-\frac{1}{3}}}{\left[1+\frac{1.646}{A}-\frac{0.191(N-Z)}{A}\right]^2}. \quad (38)$$

The deformation parameter δ is related to the well-known deformation parameter β by:

$$\delta = \frac{3}{2}\sqrt{\frac{5}{4\pi}}\beta\,. \quad (39)$$

In our calculations, the parameter β is allowed to vary in the range $-0.5 \le \beta \le 0.5$ in order to obtain a value of the reciprocal moment of inertia of a given nucleus in good agreement with the corresponding experimental value.

The single particle oscillator wave functions, needed in our computations, are taken in the form of products of three one-dimensional oscillator functions of the form [49]:

$$u_{n_x}u_{n_y}u_{n_z} = u_{n_x}(\xi)u_{n_y}(\eta)u_{n_z}(\zeta), \quad (40)$$

where:

$$u_{n_z}(\zeta) = \frac{1}{\sqrt{2^{n_z}n_z!}}\left(\frac{m\omega_z}{\pi\hbar}\right)^{\frac{1}{4}} H_{n_z}(\zeta)\exp\left(-\frac{1}{2}\zeta^2\right). \quad (41)$$

Similar equations hold for $u_{n_x}(\xi)$, and $u_{n_y}(\eta)$. In (41), $H_{n_z}(\zeta)$ is the Hermite polynomial, and the dimensionless variables are defined as:

$$(\xi,\eta,\zeta) = \left(\frac{\sqrt{m\omega_x}}{\hbar}x, \frac{\sqrt{m\omega_y}}{\hbar}y, \frac{\sqrt{m\omega_z}}{\hbar}z\right). \quad (42)$$

Using the perturbation theory, we can calculate the cranking correction to the wave function explicitly, and the result is [51]:

$$\mu_k = \Omega\sum_{j\neq k}\frac{|\langle j|\mathcal{L}|k\rangle|}{\epsilon_j-\epsilon_k}u_j, \quad (43)$$

where:

$$\mu_{n_xn_yn_z} = \mu_{n_x}(\xi)\mu_{n_y,n_z}(\eta,\zeta) = -\frac{\Omega u_{n_x}}{2\sqrt{\omega_y\omega_z}}\left\{\begin{array}{c}\sigma\sqrt{n_yn_z}u_{n_y-1}u_{n_z-1}\\ +\frac{1}{\sigma}\sqrt{n_y(n_z+1)}u_{n_y-1}u_{n_z+1}\\ +\frac{1}{\sigma}\sqrt{n_z(n_y+1)}u_{n_y+1}u_{n_z-1}\\ +\sigma\sqrt{(n_y+1)(n_z+1)}u_{n_y+1}u_{n_z+1}\end{array}\right\} \quad (44)$$

Here, μ_k is the first-order time-dependent perturbation correction for rotation about the z-axis, the functions with n_x, n_y and n_z subscripts, with arguments ξ,η and ζ, respectively, and

$$\sigma = \frac{\omega_y-\omega_z}{\omega_y+\omega_z}, \quad (45)$$

is a measure of the deformation of the potential.

4. Rigid Oscillator Moments of Inertia from Fluid Dynamical Viewpoint

We now examine the cranking moment of inertia in terms of the velocity fields. Bohr and Mottelson [76] showed that for harmonic oscillator case at the equilibrium deformation, where:

$$\frac{d}{d\delta}\sum_{i=1}\left(E_{n_x n_y n_z}\right)_i = 0, \tag{46}$$

the cranking moment of inertia is identically equal to the rigid moment of inertia:

$$\mathfrak{I}_{cr} = \mathfrak{I}_{rig} = \sum_{i=1} m\langle y_i^2 + z_i^2\rangle. \tag{47}$$

In terms of the velocity fields, this result asserts the equality of the collective kinetic energy of the Schrödinger fluid and that of rigidly rotating classical fluid:

$$\frac{m}{2}\int \rho_T \boldsymbol{v}_T \cdot (\boldsymbol{\Omega}\times\boldsymbol{r})d\tau = \frac{1}{2}\mathfrak{I}_{rig}\Omega^2 = \frac{m}{2}\int \rho_T(\boldsymbol{\Omega}\times\boldsymbol{r})^2\, d\tau, \tag{48}$$

at the equilibrium deformation. We emphasize that equation (48) holds for any number of nucleons occupying any set of single particle harmonic oscillator states at the deformation defined by equilibrium condition (46). In particular, it holds for a one particle state. For this case, equation (48) becomes:

$$\frac{m}{2}\int \rho_K \boldsymbol{v}_K \cdot (\boldsymbol{\Omega}\times\boldsymbol{r})d\tau = \frac{m}{2}\int \rho_K(\boldsymbol{\Omega}\times\boldsymbol{r})^2\, d\tau, \tag{49}$$

at the equilibrium deformation of the single particle state. Equation (49) is a remarkable identity. The scalar product of $\boldsymbol{v}_K$ and $(\boldsymbol{\Omega}\times\boldsymbol{r})$ which occurs on the left side is replaced on the right side, by the absolute square of $(\boldsymbol{\Omega}\times)$. Despite this qualitative difference between $\boldsymbol{v}_K$ and the other velocity in equation (49), this shows that, as regards their effects under the integral upon the overall kinetic energy (or the internal parameter), these two velocity fields are equivalent at the equilibrium deformation.

5. Cranking Moments and Rigid-Body Moments of Inertia

We note that the cranking moment of inertia $\mathfrak{I}_{cr}$ and the rigid-body moment of inertia $\mathfrak{I}_{rig}$ are equal only when the harmonic oscillator is at the equilibrium deformation. At other deformations, they can, and do, deviate substantially from one another. The following expressions for the cranking moment of inertia and the rigid moment of inertia $\mathfrak{I}_{rig}$ are finally given [51]:

$$\mathfrak{I}_{cr} = \frac{E}{\omega_0^2}\left(\frac{1}{6+2\sigma}\right)\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{1}{3}}\left[\sigma^2(1+q) + \frac{1}{\sigma}(1-q)\right], \tag{50}$$

$$\Im_{rig} = \frac{E}{\omega_0^2}\left(\frac{1}{6+2\sigma}\right)\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{1}{3}}[(1+q)+\sigma(1-q)], \tag{51}$$

where E is the ground-state energy:

$$E = \sum_{occ}\left[\hbar\omega_x\left(n_x + n_y + 1\right) + \hbar\omega_z\left(n_z + \frac{1}{2}\right)\right], \tag{52}$$

and q is the ratio of the summed single particle quanta in the y-and z-directions:

$$q = \frac{\sum_{occ}\left(n_y+\frac{1}{2}\right)}{\sum_{occ}\left(n_z+\frac{1}{2}\right)}. \tag{53}$$

The quantity q is known as the anisotropy of the configuration. In equations (50) and (51), the deformation of the potential, σ, is defined by [51]

$$\sigma = \frac{(\omega_y-\omega_z)}{(\omega_y+\omega_z)}. \tag{54}$$

6. The Quadrupole Moment of the Axially Deformed Nuclei

Assuming a charge distribution in accordance with the Thomas-Fermi statistical model applied to the oscillator potential, one obtains the intrinsic quadrupole moment, to the second order in the deformation parameter β [66]:

$$Q_0 = \frac{3}{\sqrt{5\pi}} ZeR^2\beta(1+0.16\beta), \tag{55}$$

where Z is the number of protons and R is to be taken equal to the radius of charge of the nucleus. The relation between the measured quadrupole moment, denoted by Q_S and Q_0 is given by [66]:

$$Q_S = \frac{3K^2-I(I+1)}{(I+1)(2I+2)} Q_0, \tag{56}$$

where I is the spin-quantum number of the specified nuclear state and K is its component along the body-fixed z-axis. It turns out that always the ground state spin of the nucleus $I_0 = \Omega = K$, where Ω is the z-component of the total angular momentum J, except when $\Omega = \frac{1}{2}$, in which case the ground state spin I_0 is given as function of the decoupling factor a, as given by Table-III of reference [66].

In the present paper, we obtained a new direct relation between the intrinsic electric quadrupole moment Q_0 and the rigid-body model moment of inertia $\Im_{rig}$ of a deformed nucleus in the following simple form:

$$Q_{0,rig} = \frac{15Z\beta(1+0.16\beta)}{2\sqrt{5\pi}AM(1+0.31\beta)} \Im_{rig}, \tag{57}$$

where M is the nucleon mass, A is the mass number, Z is the number of protons and β is the deformation parameter. The relation between the cranking moment of inertia and the rigid-body model moment of inertia can be obtained from equations (50) and (51), as follows:

$$\Im_{cr} = \frac{\sigma^2(1+q)+\frac{1}{\sigma}(1-q)}{(1+q)+\sigma(1-q)}\Im_{rig} \tag{58}$$

Accordingly, the relation between the intrinsic electric quadrupole moment and the cranking-model moment of inertia of an axially deformed nucleus can be written in the form:

$$Q_{0,cr} = \frac{15Z\beta(1+0.16\beta)\{(1+q)+\sigma(1-q)\}}{2\sqrt{5\pi}AM(1+0.31\beta)\left\{\sigma^2(1+q)+\frac{1}{\sigma}(1-q)\right\}}\Im_{cr} \tag{59}$$

Finally, the relations of the measured quadrupole moments for the two cases namely: $Q_{s,cr}$ and $Q_{s,rig}$ are easily obtained in terms of the equations (56), (57), and (59).

7. Results and Discussions

We have applied the concept of the single-particle Schrödinger fluid to the ten nuclei in the p-shell, namely the nuclei: ^{6}Li, ^{7}Li, ^{8}Li, ^{9}Li, ^{10}Li, ^{11}Li, ^{8}Be, ^{10}B, ^{11}B, and ^{12}C in order to calculate the cranking-model, the rigid-body model, and the equilibrium-model moments of inertia of these nuclei as functions of their deformation parameters β. Accordingly, the intrinsic electric quadrupole moments of these nuclei are obtained from our new derived formulas and therefore, the measured quadrupole moments of the ten nuclei are then obtained.

In Figures 1-10**,** we present the variations of the calculated values of the reciprocal moments of inertia according to the cranking model and the rigid-body model with respect to the deformation parameter β for the nuclei ^{6}Li, ^{7}Li, ^{8}Li, ^{9}Li, ^{10}Li, ^{11}Li, ^{8}Be, ^{10}B, ^{11}B, and ^{12}C, respectively.

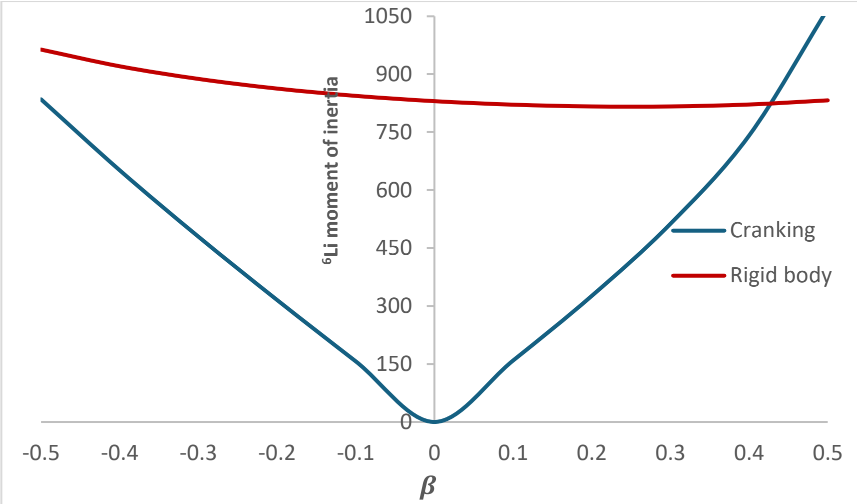

Fig. 1 Reciprocal moments of inertia of the nucleus ^{6}Li.

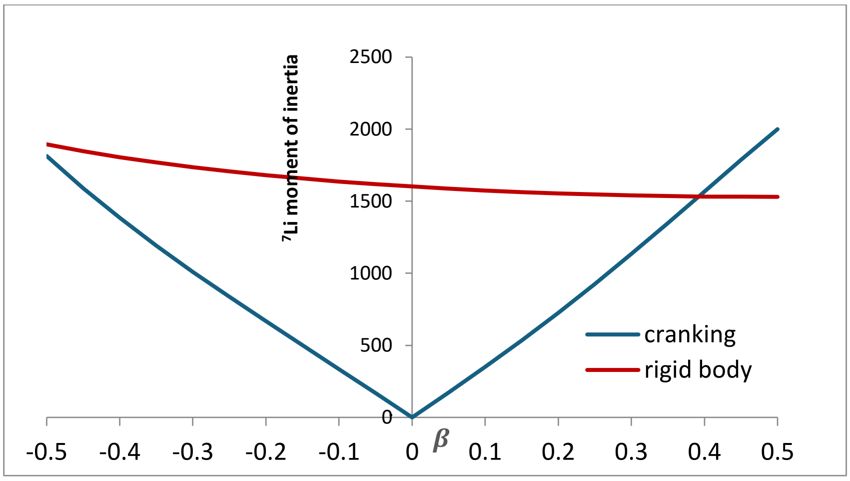

Fig. 2 Reciprocal moments of inertia of the nucleus ^{7}Li

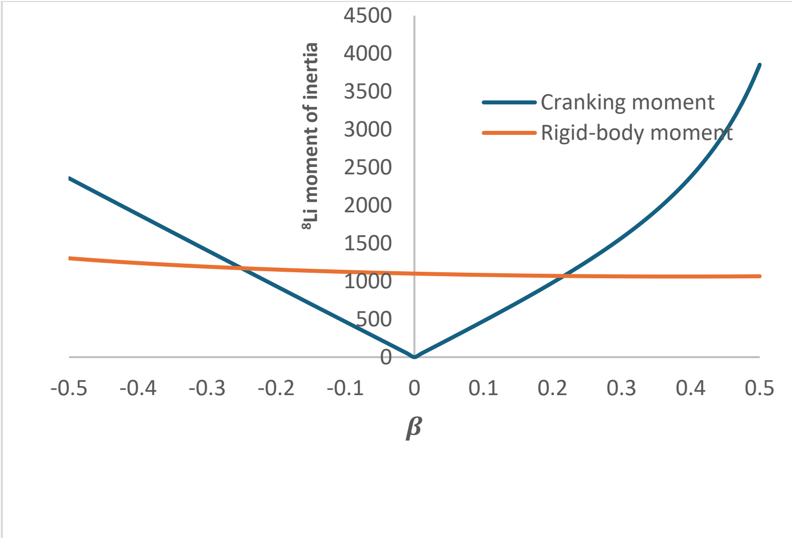

Fig. 3 Reciprocal moments of inertia of the nucleus ^{8}Li

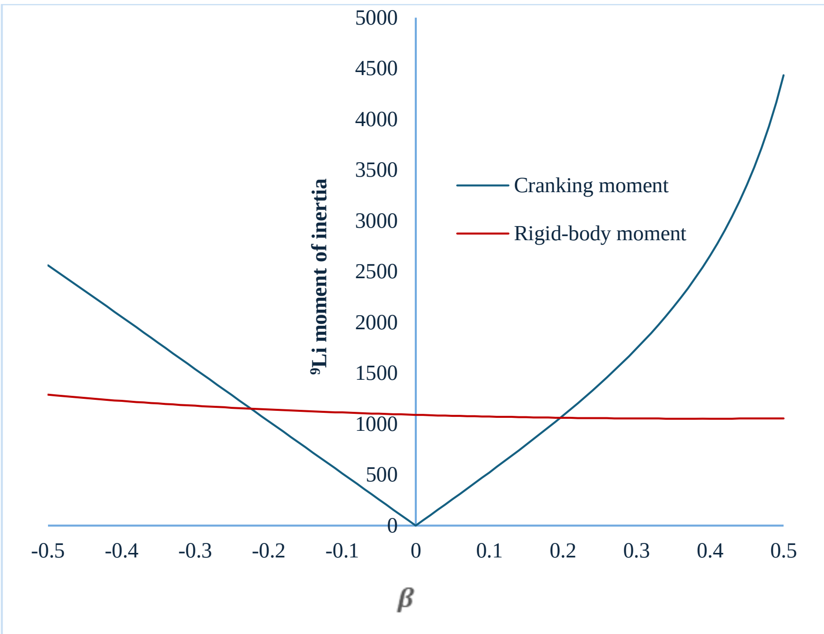

Fig. 4 Reciprocal moments of inertia of the nucleus ^{9}Li

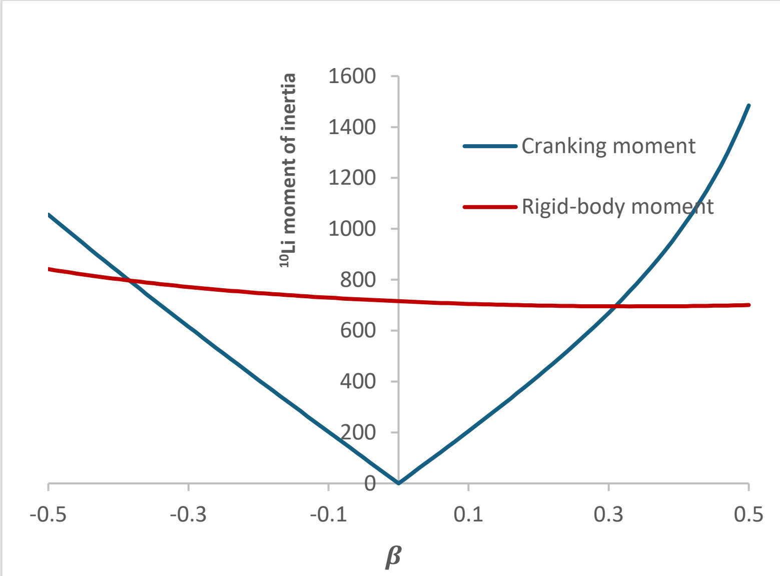

Fig. 5 Reciprocal moments of inertia of the nucleus ^{10}Li

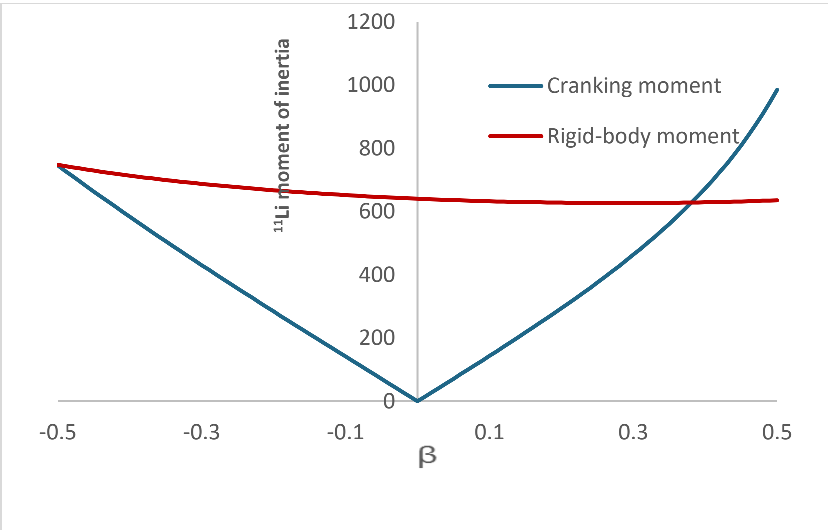

Fig. 6 Reciprocal moments of inertia of the nucleus ^{11}Li

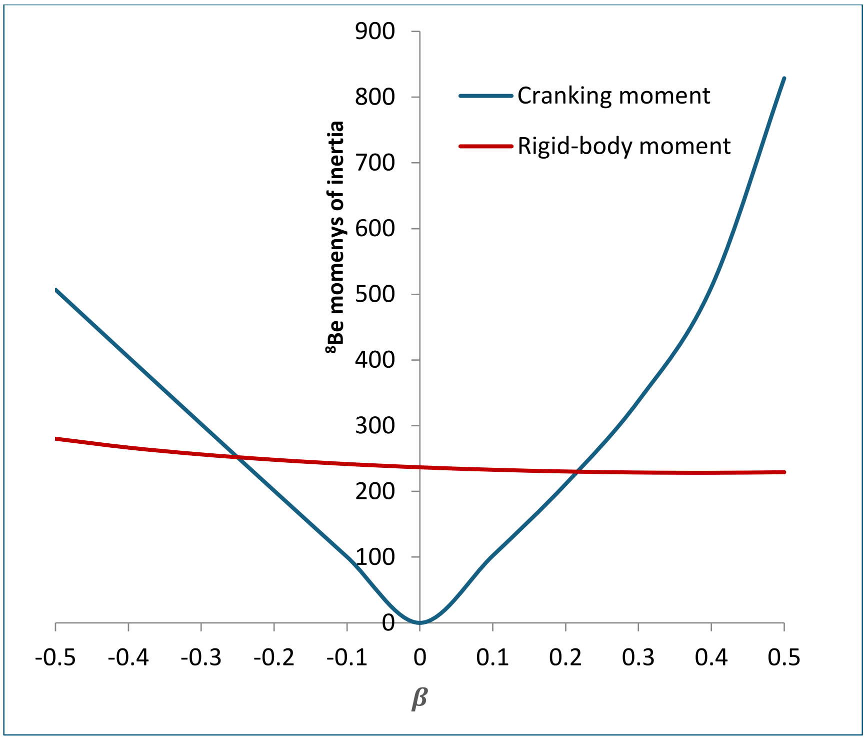

Fig. 7 Reciprocal moments of inertia of the nucleus ^{8}Be.

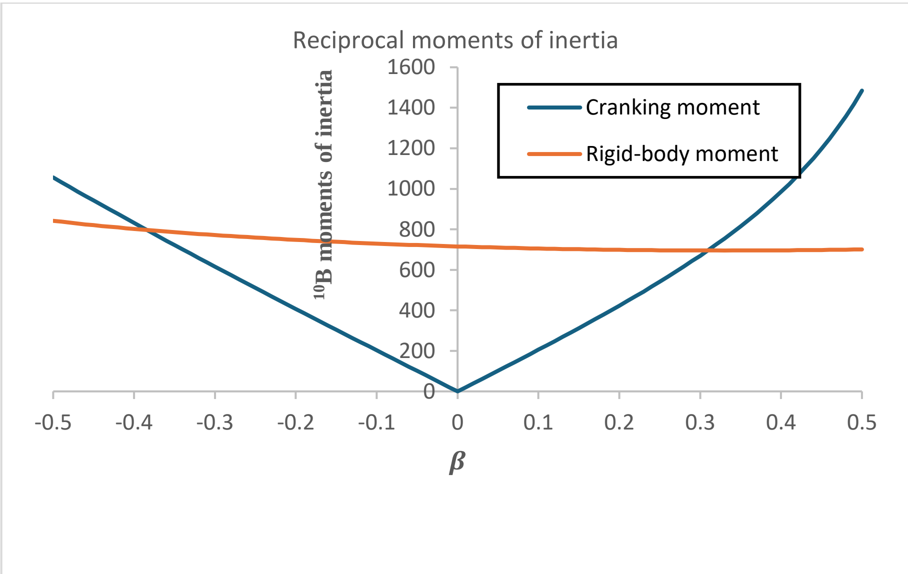

Fig. 8 Reciprocal moments of inertia of the nucleus ^{10}B

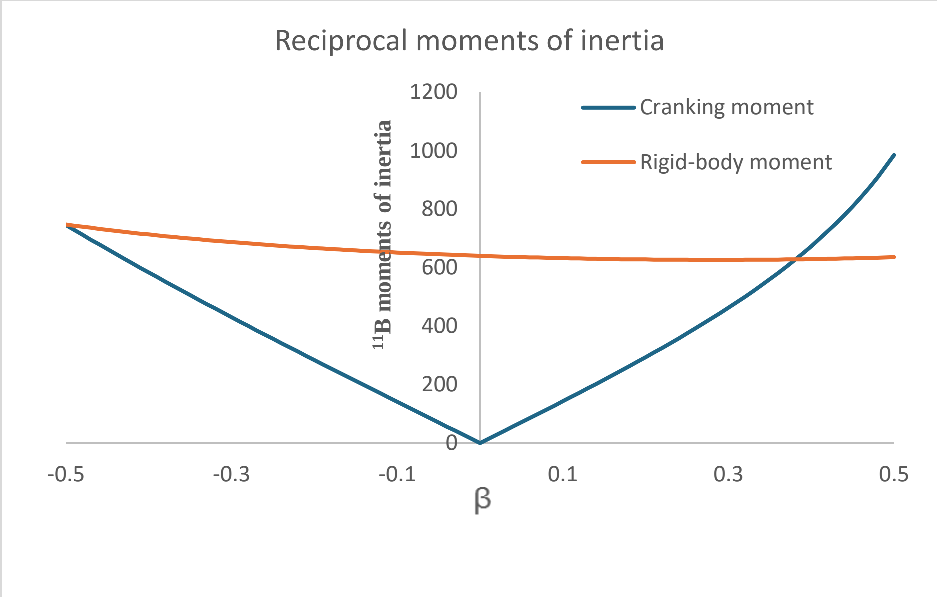

Fig. 9 Reciprocal moments of inertia of the nucleus ^{11}B

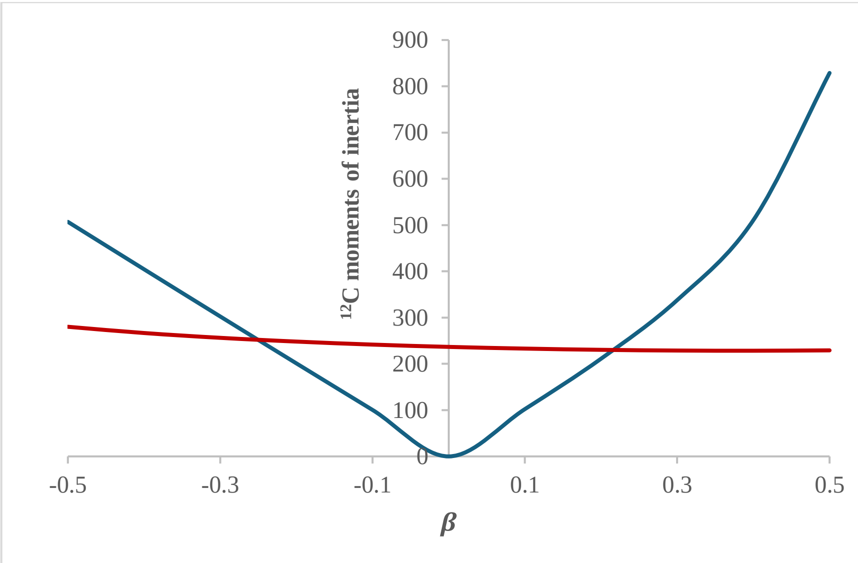

Fig. 10 Reciprocal moments of inertia of the nucleus ^{12}C. Blue line for cranking moment of inertia and dark red for rigid-body moment of inertia.

In Table-1, we present the calculated values of the reciprocal moments of inertia of the ten nuclei in the p-shell, namely: ^{6}Li, ^{7}Li, ^{8}Li, ^{9}Li, ^{10}Li, ^{11}Li, ^{8}Be, ^{10}B, ^{11}B, and ^{12}C, according to the cranking model, $\mathfrak{I}_{cr}$, and the rigid body model, $\mathfrak{I}_{rig}$, together with the values of the deformation parameter β and the none-deformed oscillator parameter $\hbar\omega_0{}^0$. In Table-1, we present also the experimental values of the reciprocal moments of inertia of these nuclei, obtained from the low-lying rotational spectra of these nuclei [74,81].

In Table-2, we present the values of the reciprocal equilibrium moments of inertia, $\Im_{equ}$, for the ten nuclei in the p-shell, namely: ^{6}Li, ^{7}Li, ^{8}Li, ^{9}Li, ^{10}Li, ^{11}Li, ^{8}Be, ^{10}B, ^{11}B, and ^{12}C together with the corresponding experimental values and the values of the deformation parameter, β, at which the cranking model and the rigid body model moments of inertia are equal. The values of the non-deformed oscillator parameter $\hbar\omega_0{}^0$ are also given in Table-2.

Table-1 The calculated values of the reciprocal moments of inertia of the nuclei ^{6}Li, ^{7}Li, ^{8}Li, ^{9}Li, ^{10}L, ^{11}Li, ^{8}Be, ^{10}B, ^{11}B, and ^{12}C which are in good agreement with the corresponding experimental values are given. The values of $\hbar\omega_0^0$, β, and the corresponding experimental values [74,81] are also given in this table.

Nucleus	β	$\hbar\omega_0^0$ MeV	$\frac{\hbar^2}{2\Im_{crank}}$ KeV	$\frac{\hbar^2}{2\Im_{rigid}}$ KeV	$\frac{\hbar^2}{2\Im_{exper}}$KeV [74,81]
^{6}Li	0.26	9.594	493.44	716.04	500
^{7}Li	0.21	11.543	687.14	1588.34	650.00
^{8}Li	0.16	13.21	773.01	1075.15	750 .00
^{9}Li	0.17	14.14	909.16	1064.51	900.00
^{10}Li	0.31	12.02	697.41	695.55	715.72
^{11}Li	0.20	12.77	293.90	627.87	297.71
^{8}Be	0.277	11.36	446.5	282.6	446.19
^{10}B	-0.35 0.31	12.02	722.63 697.41	786.12 695.55	715.72
^{11}B	-0.21 0.20	12.77	298.20 293.90	668.94 627.87	297.71
^{12}C	-0.434	12.24	742.4	212.3	750.0

Table-2 Reciprocal equilibrium moments of inertia, $\mathfrak{I}_{equ}$, for the nuclei ^{6}Li, ^{7}Li, ^{8}Li, ^{9}Li, ^{10}Li, ^{11}Li, ^{8}Be, ^{10}B, ^{11}B, and ^{12}C together with the corresponding experimental values, the values of the deformation parameter, β, at which the cranking model and the rigid body model moments of inertia are equal and the values of the parameter $\hbar\omega_0{}^0$.

Nucleus	β	$\hbar\omega_0^0$ (MeV)	$\frac{\hbar^2}{2\mathfrak{I}_{equ}}$ (MeV)	$\frac{\hbar^2}{2\mathfrak{I}_{exp}}$ (MeV) [74,81]
^{6}Li	-0.33	9.594	1642.71	500.00
^{7}Li	0.39	11.796	1533.55	539.86
^{8}Li	0.21	13.208	1069.55	750.00
^{9}Li	0.19	14.14	1062.437	900.00
^{10}Li	0.32	12.02	695.480	715.72
^{11}Li	0.37	12.77	627.991	297.71
^{8}Be	0.20	11.36	889.96	500-900
^{10}B	0.31	12.02	695.60	715.75
^{11}B	0.38	12.77	848.48	500-750
^{12}C	-0.20	12.24	550.87	418.64

We have used the two new formulas (57) and (59) to calculate the electric quadrupole moments $Q_{s,cr}$ and $Q_{s,rig}$ of the nuclei ^{6}Li, ^{7}Li, ^{8}Li, ^{8}Be, ^{9}Be, ^{10}B, ^{11}B, and ^{12}C as functions of the deformation parameter β and the non-deformed oscillator parameter $\hbar\omega_0^0$. As an illustration, we present in Figs. 11, and 12, the variation of the electric quadrupole moments of the two nuclei ^{10}B and ^{11}B with respect to the deformation parameter β.

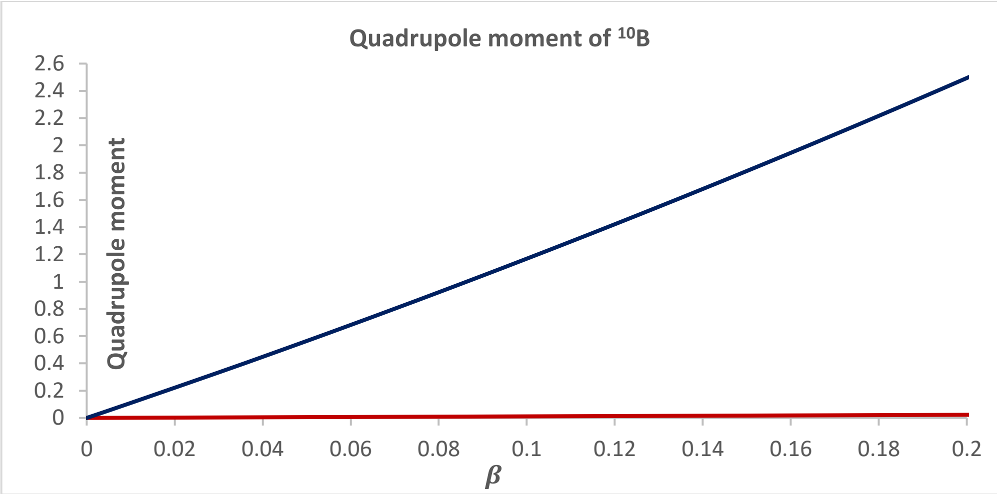

Fig. 11 Variation of the electric quadrupole moments of ^{10}B with respect to the deformation parameter β. Blue line for moment derived from cranking moment of inertia and dark-red line for moment derived from rigid-body moment of inertia.

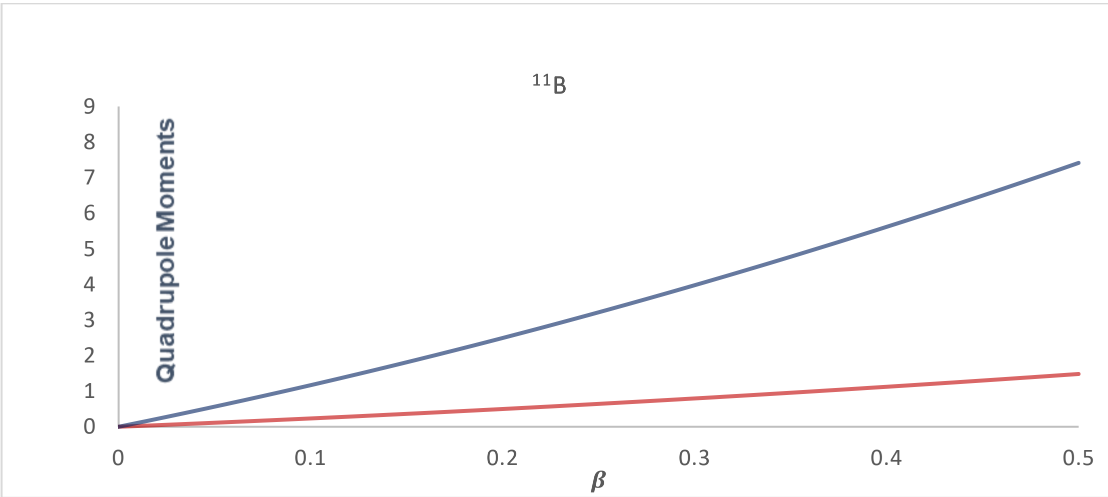

Fig. 12 Variation of the electric quadrupole moments of ^{11}B with respect to the deformation parameter β. Blue line for moment derived from cranking moment of inertia and dark-red line for moment derived from rigid-body moment of inertia.

In Table-3, we present the calculated values of the electric quadrupole moments of the nuclei ^{6}Li, ^{7}Li, ^{8}Li, ^{8}Be, ^{9}Be, ^{10}B, ^{11}B, and ^{12}C. The values of the total spin and parity I^{π} of these nuclei are also given in Table-3. Furthermore, the values of the deformation parameter β and the none-deformed oscillator parameter $\hbar\omega_0^0$, for which the calculated values of the electric quadrupole moments are in good agreement with the corresponding experimental values, are also given in this table.

Table-3 Quadrupole moments of the nuclei ^{6}Li, ^{7}Li, ^{8}Li, ^{8}Be, ^{9}Be, ^{10}B, ^{11}B, and ^{12}C.

Nucleus	β	I^π	$\hbar\omega_0^0$ (MeV)	Q_s crank (e m barns)	Q_s rigid (e m barns)	Q_s exp [66,81] (e m barns)
^{6}Li	-0.12	1^+	9.594	-0.079	-0.051	-0.083
^{7}Li	-0.18	$\frac{3}{2}^-$	11.796	-3.574	-1.785	-4.060
^{8}Li	0.24	2^+	13.208	3.652	3.988	3.170
^{8}Be	0.277	0^+	11.36	-0.022	-0.048	N/A
^{9}Be	0.26	$\frac{3}{2}^-$	12.561	5.022	5.876	5.300
^{10}B	0.34	3^+	12.022	6.432	6.864	8.472
^{11}B	0.37	$\frac{3}{2}^-$	12.768	3.879	4.393	4.065
^{12}C	0.18	2^+	12.238	5.871	6.333	6.000

It is seen from Table-1 that the calculated values of the moments of inertia of the considered nuclei according to the cranking model by using the concept of the single-particle Schrödinger fluid are in good agreement with the corresponding experimental values, a result which shows that the concept of this fluid is reliable and can be applied successfully to other deformed nuclei. Moreover, we see from Table-1 that the values of the deformation parameter β for the considered nuclei decrease for nuclei, which are close to a closed shell, a result that agrees with the experimental findings. Moreover, it is seen from Table-1 that, according to the calculations, the considered deformed nuclei may have oblate deformations (negative values of β) or prolate deformations (positive values of β).

Furthermore, it is seen from Table-2 that the calculated values of the reciprocal equilibrium moments of inertia for the lithium-isotopes are not in good agreement with the corresponding experimental values, a result that is expected, because the pairing correlation is very important for such light nuclei [45-47].

The analysis of the quadrupole moments of the considered nuclei shows that, among all the considered nuclei, the nuclei ^{6}Li and ^{7}Li have, only, oblate shapes while the others have prolate deformations. Moreover, it is seen from Table-3 that the new formulas, which relates the quadrupole moment of a deformed nucleus to its moment of inertia, can be used successfully for the calculations of the quadrupole moment of other axially deformed nuclei.

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Dr. S. B. Doma, Professor of Applied Mathematics and Theoretical Physics, Faculty of Science. Alexandria University, Egypt, 1) Member of the Editorial Board of the Open Applied Mathematics Journal.

2) The Editor-in-Chief of Alexandria Journal of Mathematics: www.alexjournals.org/mathematics

3) The Vice Editor-in-Chief of Alexandria Journal of Physics www.alexjournals.org/physics

4) Member of the Editorial Board of the Bioinfo Publications Journals.

5) Member of the Editorial Board of the Scirea Journal of Physics.

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