Alexandria Journal of Physics (ISSN 2090-4789)
Volume1 -Number1 - March 2011

Analytical Expressions for Some Outer Product Isoscalar Factors

M. M. Hammad

Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt.
E-mail address quark_su3@yahoo.com

Abstract

The linear equation method is modified to be used in the calculations of the outer product coupling coefficients (OPCCs) for the symmetric group. The classification of the interaction operators with respect to the permutation symmetry of the n particle system is given with the relation of the OPCCs of the symmetric group S_n. We derived all the required outer product isoscalar factors (OPISFs) for one, two and three body interaction operators. As examples, we have calculated the OPISFs of $[1]\otimes[n-1]$, $[2]\otimes[n-2]$, $\left[1^2\right]\otimes[n-2]$, $[3]\otimes[n-3]$, $[21]\otimes[n-3]$, and $\left[1^3\right]\otimes[n-3]$.The general method of finding the modified ground representation (MGR) of the generators of the symmetric groups is discussed.

AMS Classifications: 20C35, 20B30, 20B35, 20C40.

Keywords: Applications of group representations to physics, Symmetric groups, Subgroups of symmetric groups, Computational methods, Outer products, Isoscalar factors.

1 Introduction

The symmetric group S_n finds a place in the study of systems consisting of n identical particles as a part of the symmetry group of the Hamiltonian. There may, in general, be systems consisting of two sets of n_1 and n_2 particles, each in a state which transforms according to an irreducible representation (IRREP) $[\lambda_1]$ of S_{n_1} and IRREP $[\lambda_2]$ of S_{n_2}. These states may be two different shells in an atomic or nuclear shell model or two different clusters in a nuclear cluster model. The question posed here is how to construct a state of the total system with the permutation symmetry $[\lambda]$. The outer product coubling coefficients (OPCCs) achieve this transformation [1]. The OPCCs are in fact closely related to the subduction coefficients, which were introduced to describe the states of a physical system with n identical particles as composed of two subsystems with n_1 and n_2 particles, respectively $(n_1+n_2=n)$ [2-4].

It is well known that the OPCCs can be factorized into successive products of isoscalar factors (Racah's factorization lemma) [5]. So that the outer product isoscalar factors (OPISFs) of S_n are required.

Many techniques have been given for calculating OPCCs. The investigation of these coefficients is incomplete. Several issues are of particular interest. Firstly: the recursion formulae are deduced for the construction of the OPCCs for the symmetric group using double coset technique and closed algebraic formulae exist for some cases [6-10]. Secondly: OPCCs up to and for S_6 were produced by Chen *et al* based on the eigenfuction method [5].

OPCCs or OPISFs of the symmetric groups calculated by any of these methods quickly become intractable with increasing rank n. They can only provide tables of calculated OPCCs and OPISFs rather than a direct computational formula from which OPCCs and OPISFs for the related group can be conveniently evaluated, as one always desires such a formula for simplifying the numerical calculation programmes of quantum mechanics and chemistry.

Linear equation method [11-15] is used to evaluate Clebsch-Gordan Coefficients and subduction coefficients of the symmetric groups. In this paper the linear equation method is modified to be used in the calculation of the OPCCs for the symmetric group. So that, the general method to find the modified ground representations (MGRs) of the generators is discussed.

2 Background of the Symmetric Group

A partition of a positive integer n is a set of integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. A Young diagram is a diagram with n boxes arranged in n rows corresponding to a partition of n. A standard Young tableau is a Young diagram in which the n boxes have been filled with the numbers $1,\ 2,\ \ldots,\ n$, each number used exactly once, and the numbers appear in an ascending order within each row or column from left to right and top to bottom. Each standard tableau can be written in a compact way by Yamanouchi symbol $(\ r_n \ldots r_1)$ [16-17]. This is an array of n numbers, the r_i being the rows in the standard tableau in which the number i appears. The nonequivalent IRREPs of S_n are defined by the different partitions of the number n into positive integral components. Each IRREP is typified by one such partition. The IRREPs of S_n are therefore usually enumerated by the symbol $[\lambda]$ for the corresponding Young diagrams, and denoted by $[\lambda]$. In general an IRREP $[\lambda]$ of the group S_n becomes reducible upon passing to the subgroup S_{n-1}, and can be decomposed into IRREPs $\left[\acute{\lambda}\right]$ of this subgroup. The representations $\left[\acute{\lambda}\right]$ into which $[\lambda]$ decomposes are determined by all the Young diagrams with $n-1$ boxes that are obtained from the initial diagram upon removing one of its boxes. It is also well known that we can label the standard or canonical Young-Yamanouchi basis for $[\lambda]$, which is adapted to the chain $S_n \supset S_{n-1} \supset\ \ldots\ \supset S_2 \supset S_1$, by $\left| \begin{matrix} [\lambda] \\ m \end{matrix} \right\rangle$, where m designates either a Young tableau or a Yamanouchi symbol $(\ r_n \ldots r_1)$. The dimension of the IRREPs is given by the number of standard Young tableaux and is denoted by $d_{[\lambda]}$.

3 Outer Product Coupling Coefficient

Consider groups S_{n_1} and S_{n_2}, the elements of which operate, respectively, on n_1 objects $\{1,\ 2,\ \ldots, n_1\} = \omega_0^1$ and n_2 objects $\{n_1+1, n_1+2,\ \ldots, n_1+n_2\} = \omega_0^2$. If $[\lambda_1]$ and $[\lambda_2]$ are the IRREPs of the symmetric groups S_{n_1} and S_{n_2}, respectively, the outer product $[\lambda_1]\otimes[\lambda_2]$ is a representation, in general reducible, of the symmetric group on n_1+n_2 objects. The decomposition of outer product representation into IRREPs of $S_{n_1+n_2}$:

$$[\lambda_1] \otimes [\lambda_2] = \sum_{[\lambda]} \{\lambda_1\lambda_2\lambda\} [\lambda] , \tag{3.1}$$

where the integers $\{\lambda_1\lambda_2\lambda\}$ are the number of times $[\lambda]$ occurs in the outer product representation, is determined by the Littlewood rule[5].
Let the Yamanouchi basis of the permutation group $S_{n_1}(\omega_0^1)$ and $S_{n_2}(\omega_0^2)$ be denoted as follows

$$\left|Y_{m_1}^{[\lambda_1]}\left(\omega_0^1\right)\right\rangle = \left|\begin{array}{c}[\lambda_1]\\ m_1\omega_0^1\end{array}\right\rangle , \left|Y_{m_2}^{[\lambda_2]}\left(\omega_0^2\right)\right\rangle = \left|\begin{array}{c}[\lambda_2]\\ m_2\omega_0^2\end{array}\right\rangle , \tag{3.2}$$

$$m_1 = 1,\ 2,\ \ldots, d_{[\lambda_1]}, \qquad m_2 = 1,\ 2,\ \ldots, d_{[\lambda_2]},$$

Their products are denoted by

$$\left|\begin{array}{ccc}[\lambda_1] & [\lambda_2] & \\ m_1 & m_2 & \omega_0\end{array}\right\rangle = \left|Y_{m_1}^{[\lambda_1]}\left(\omega_0^1\right) Y_{m_2}^{[\lambda_2]}\left(\omega_0^2\right)\right\rangle , \tag{3.3}$$

$$\omega_0 = \left\{\omega_0^1, \omega_0^2\right\} = \{1,\ 2,\ \ldots, n\} .$$

Let us now replace the index sets ω_0^1 and ω_0^2 by $\omega^1 = \{j_1,\ \ldots j_{n_1}\}$, $j_1 < \cdots < j_{n_2}$, and $\omega^2 = \{j_{n_1+1},\ \ldots j_{n_1+n_2}\}$, $j_{n_1+1} < \ldots < j_{n_1+n_2}$, and define $d = \binom{n}{n_1}$ - normal order sequences, $\omega = \left\{\omega^1, \omega^2\right\}$, such that $\omega^1 \cup \omega^2 = \omega_0 = \{1,\ 2,\ \ldots, n\}$, $\omega^1 \cap \omega^2 = \emptyset$, and introduce the following map

$$\begin{pmatrix}\omega_0\\ \omega\end{pmatrix} = \begin{pmatrix}\omega_0^1 & \omega_0^2\\ \omega^1 & \omega^2\end{pmatrix} = Q_\omega , \tag{3.4}$$

The left coset decomposition of S_n with respect to the subgroup $S_{n_1} \otimes S_{n_2}$ is denoted by

$$S_n = \sum_{\omega=1}^{d} Q_\omega (S_{n_1} \otimes S_{n_2}). \tag{3.5}$$

Applying the $d\ - Q_\omega^{'}$s to (3.3) we get altogether $N = dd_{[\lambda_1]}d_{[\lambda_2]}$ basis vectors

$$Q_\omega \left|\begin{array}{ccc}[\lambda_1] & [\lambda_2] & \\ m_1 & m_2 & \omega_0\end{array}\right\rangle = \left|\begin{array}{ccc}[\lambda_1] & [\lambda_2] & \\ m_1 & m_2 & \omega\end{array}\right\rangle = \left|Y_{m_1}^{[\lambda_1]}\left(\omega^1\right) Y_{m_2}^{[\lambda_2]}\left(\omega^2\right)\right\rangle . \tag{3.6}$$

In the following we assume that

$$\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] & \\ \acute{m}_1 & \acute{m}_2 & \acute{\omega} \end{matrix} \middle| \begin{matrix} [\lambda_1] & [\lambda_2] & \\ m_1 & m_2 & \omega \end{matrix} \right\rangle = \delta_{m_1 \acute{m}_1} \delta_{m_2 \acute{m}_2} \delta_{\omega \acute{\omega}}, \tag{3.7}$$

is true, that is, the N basis vectors are orthogonal.

The N basis vectors of (3.6) can be combined linearly into the irreducible basis of S_n

$$\left| \begin{matrix} [\lambda]\,\alpha \\ m \end{matrix} \right\rangle = \sum_{m_1 m_2 \omega} \left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m \end{matrix} \right) \left| \begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \right\rangle, \tag{3.8}$$

where the coefficients $\left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m \end{matrix} \right)$ are called the OPCCs.

The OPCCs satisfy the unitary conditions

$$\sum_{m_1 m_2 \omega} \left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m \end{matrix} \right) \left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \middle| \begin{matrix} [\acute{\lambda}]\,\acute{\alpha} \\ \acute{m} \end{matrix} \right) = \delta_{[\lambda][\acute{\lambda}]} \delta_{\alpha\acute{\alpha}} \delta_{m\acute{m}}, \tag{3.9}$$

$$\sum_{[\lambda]\alpha m} \left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m \end{matrix} \right) \left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ \acute{m}_1\acute{\omega}^1 & \acute{m}_2\acute{\omega}^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m \end{matrix} \right) = \delta_{m_1 \acute{m}_1} \delta_{m_2 \acute{m}_2} \delta_{\omega \acute{\omega}}, \tag{3.10}$$

and the following symmetry property

$$\left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m \end{matrix} \right) = \varepsilon \left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ m_2\omega^2 & m_1\omega^1 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m \end{matrix} \right), \tag{3.11}$$

where ε is a phase factor.
The overall phase is fixed by requiring that the OPCC with $m_1 = m_2 = m = 1$ and with the smallest possible index ω be positive [5]

$$\left(\begin{matrix} [\lambda_1] & [\lambda_2] \\ 1\omega^1 & 1\omega^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ 1 \end{matrix} \right) \Bigg|_{\omega_{min}} > 0. \tag{3.12}$$

4 The linear Equation Method

The symmetric group S_n can be defined by $n-1$ generators $\{G_i : i = 1, ...n-1\}$,

which are adjacent permutations that satisfy the following relations $[11-15]$:

$$G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1},$$

$$G_iG_j = G_jG_i \text{ for } |i-j| \geq 2,$$

$$G_i^2 = 1. \tag{4.1}$$

Given a standard Young tableau m, we define the action $G_i(m)$ in the following way: let $G_i(m)$ be the Young tableau obtained by interchanging the numbers i and $i+1$ in m. It is understood that if the resultant tableau is not a standard one, the corresponding basis vector $\left| \begin{matrix} [\lambda] \\ G_i(m) \end{matrix} \right\rangle$ is set to zero. The G_i acts on the standard basis vectors $\left| \begin{matrix} [\lambda] \\ m \end{matrix} \right\rangle$ of the IRREP $[\lambda]$ as follows [2]:

$$G_i \left| \begin{matrix} [\lambda] \\ m \end{matrix} \right\rangle = \frac{1}{\sigma_i} \left| \begin{matrix} [\lambda] \\ m \end{matrix} \right\rangle + \sqrt{1-\frac{1}{\sigma_i^2}} \left| \begin{matrix} [\lambda] \\ G_i(m) \end{matrix} \right\rangle, \tag{4.2}$$

where σ_i is the axial distance from the box i to the box $i+1$ in (m).

Applying the action G_i with $i = 1, ...n-1$ to (3.8), since the product basis vectors are linearly independent, we have

$$\sum_{\acute{m}\acute{\omega}} \left\langle \begin{matrix} [\lambda]\,\alpha \\ \acute{m}\acute{\omega} \end{matrix} \right| G_i \left| \begin{matrix} [\lambda]\,\alpha \\ m\omega \end{matrix} \right\rangle \left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ \acute{m}_1\acute{\omega}^1 & \acute{m}_2\acute{\omega}^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m\omega \end{matrix} \right\rangle$$
$$= \sum_{m_1m_2\omega^1\omega^2} \left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \middle| \begin{matrix} [\lambda]\,\alpha \\ m\omega \end{matrix} \right\rangle$$
$$\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ \acute{m}_1\acute{\omega}^1 & \acute{m}_2\acute{\omega}^2 \end{matrix} \right| G_i \left| \begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \right\rangle \quad , \tag{4.3}$$

where $\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ \acute{m}_1\acute{\omega}^1 & \acute{m}_2\acute{\omega}^2 \end{matrix} \right| G_i \left| \begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \right\rangle$ is the matrix element of the generator G_i of S_n in the uncoupling basis

$$\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ \acute{m}_1\acute{\omega}^1 & \acute{m}_2\acute{\omega}^2 \end{matrix} \right| G_i \left| \begin{matrix} [\lambda_1] & [\lambda_2] \\ m_1\omega^1 & m_2\omega^2 \end{matrix} \right\rangle$$
$$= \xi_{\omega\acute{\omega}} \left\langle \begin{matrix} [\lambda_1] \\ \acute{m}_1\omega_0^1 \end{matrix} \right| p_1 \left| \begin{matrix} [\lambda_1] \\ m_1\omega_0^1 \end{matrix} \right\rangle \left\langle \begin{matrix} [\lambda_2] \\ \acute{m}_2\omega_0^2 \end{matrix} \right| p_2 \left| \begin{matrix} [\lambda_2] \\ m_2\omega_0^2 \end{matrix} \right\rangle, \tag{4.4}$$

and

$$\xi_{\omega\acute{\omega}} = \begin{cases} 1 & if\ Q_{\acute{\omega}}^{-1}G_iQ_\omega = p_1p_2 \in S_{n_1} \otimes S_{n_2} \\ 0 & otherwise, \end{cases} \quad , \tag{4.5}$$

where $p_1 \in S_{n_1}$ and $p_2 \in S_{n_2}$.

Having found the matrix elements of G_i, in the uncoupling basis, we can use (4.3) to obtain a recursion formula for OPCCs. It is convenient to introduce the MGR $\Im_{\omega\acute{\omega}}(G_i)$ as

$$\Im_{\omega\acute{\omega}}(G_i) = \begin{cases} p_1p_2 & if\ Q_{\acute{\omega}}^{-1}G_iQ_\omega = p_1p_2 \in S_{n_1} \otimes S_{n_2} \\ 0 & otherwise. \end{cases} \quad . \tag{4.6}$$

Theorem

The MGRs of the generators $G_i = (i, i+1)$ of the symmetric group $S_{n_1+n_2}$ with recpect to its subgroups $S_{n_1} \otimes S_{n_2}$, can be reduced into the following

$$\Im_{\omega\acute{\omega}}(i,i+1) = \begin{cases} (xy)\,E_2 & if\ i\ and\ i+1 \in \omega^1, where\ (xy) \in S_{n_1}, \\ E_1\,(xy) & if\ i\ and\ i+1 \in \omega^2, where\ (xy) \in S_{n_2}, \\ E_1 E_2 & if\ i\ \in \omega^1 and\ i+1 \in \omega^2 or\ i \in \omega^2\ and\ i+1 \in \omega^1, \end{cases} \quad (4.7)$$

where x and y are the positions of i and $i+1$,respectively, in the order sequance ω and $E_1 \in S_{n_1}, E_2 \in S_{n_2}$.

Proof

Let us consider the order sequence

$$\omega = \{j_1, ...j_{n_1}, j_{n_1+1}, ...j_{n_1+n_2}\}, j_1 < ... < j_{n_1}, j_{n_1+1} < ... < j_{n_1+n_2}, \quad (4.8)$$

therefore $Q_\omega = \begin{pmatrix} 1 & ... & n_1 & n_1+1 & ... & n_1+n_2 \\ j_1 & ... & j_{n_1} & j_{n_1+1} & ... & j_{n_1+n_2} \end{pmatrix}$, and each one of the j_t represents any one of the numbers $1, 2, ...n_1+n_2$. Similarly,

$$\acute{\omega} = \{l_1, ...l_{n_1}, l_{n_1+1}, ...l_{n_1+n_2}\}, l_1 < ... < l_{n_1}, l_{n_1+1} < ... < l_{n_1+n_2}, \quad (4.9)$$

therefore $Q_{\acute{\omega}} = \begin{pmatrix} 1 & ... & n_1 & n_1+1 & ... & n_1+n_2 \\ l_1 & ... & l_{n_1} & l_{n_1+1} & ... & l_{n_1+n_2} \end{pmatrix}$, and each one of the l_t represents any one of the numbers $1, 2, ...n_1+n_2$.

Case (1)

If i and $i+1 \in \omega^1$.

In this case we have

$$Q_{\acute{\omega}}^{-1}(i,i+1)\,Q_\omega = \begin{pmatrix} 1 & ... & x & ... & y & ... & n_1 & n_1+1 & ... & n_1+n_2 \\ i_1 & ... & i_y & ... & i_x & ... & i_{n_1} & i_{n_1+1} & ... & i_{n_1+n_2} \end{pmatrix}, \quad (4.10)$$

where $i_1, ..., i_{n_1+n_2}$ are the positions of $j_1, ..., j_{n_1+n_2}$ in the order sequence $\acute{\omega}$, respectively. In order to write $Q_{\acute{\omega}}^{-1}(i,i+1)\,Q_\omega$ such that

$$Q_{\acute{\omega}}^{-1}(i,i+1)\,Q_\omega = p_1 p_2 \in S_{n_1} \otimes S_{n_2}, \quad (4.11)$$

i.e. $i_1, .., i_{n_1} \in \{1, ...n_1\}$ and $i_{n_1+1}, ..., i_{n_1+n_2} \in \{n_1+1, ...n_1+n_2\}$, the positions of $j_1...j_{n_1}$ (in the order sequance $\acute{\omega}$) must take any permutation of $\{1, ...n_1\}$ and the positions of $j_{n_1+1}, ..., j_{n_1+n_2}$ must take any permutation of $\{n_1+1, ...n_1+n_2\}$. But j_t satisfy (4.8), therefore, $\acute{\omega} = \{j_1, ...j_{n_1+n_2}\}$ and the order sequence $\acute{\omega}$ satisfies (4.11), if and only if $\acute{\omega} = \omega$, and the ground represntation looks like the regular representation in that there is only one non-vanishing matrix element in each row or column and in this case from (4.10), we can write $Q_{\acute{\omega}}^{-1}(i,i+1)\,Q_\omega = (xy)\,E_2, where\ (xy) \in S_{n_1}, E_2 \in S_{n_2}$. Similarly, if i and $i+1 \in \omega^2$, we have $Q_{\acute{\omega}}^{-1}(i,i+1)\,Q_\omega = E_1\,(xy)\,, where\ E_1 \in S_{n_1}, (xy) \in S_{n_2}$.

Case (2)

If $i \in \omega^1$ and $i+1 \in \omega^2$ or $i \in \omega^2$ and $i+1 \in \omega^1$, it is easy to prove that $\acute{\omega} = (xy)\,\omega$, and $Q_{\acute{\omega}}^{-1}\,(i,i+1)\,Q_\omega = E_1E_2, where\ E_1 \in S_{n_1}, E_2 \in S_{n_2}$.

The MGRs of the generators of the group S_4 with recpect to its subgroups $S_2{\otimes}S_2$, are listed in Table (1). Since the ground representation contains only one non-vanishing matrix element in each row or column,Table (1) contains for every ω the corresponding $\acute{\omega}$ and the non-vanishing matrix element $Q_{\acute{\omega}}^{-1}\,(i,i+1)\,Q_\omega$ only.

Table (1) The MGRs of the generators of the symmetric group S_4 with respect to its subgroups $S_2 \otimes S_2$

$\acute{\omega}$	ω	$Q_{\acute{\omega}}^{-1}\,(12)\,Q_\omega$
12.34	12.34	$(12)\,E$
34.12	34.12	$E\,(34)$
13.24	23.14	EE
24.13	14.23	EE
23.14	13.24	EE
14.23	24.13	EE

$\acute{\omega}$	ω	$Q_{\acute{\omega}}^{-1}\,(23)\,Q_\omega$
13.24	12.34	EE
24.13	34.12	EE
23.14	23.14	$(12)\,E$
14.23	14.23	$E\,(34)$
12.34	13.24	EE
34.12	24.13	EE

$\acute{\omega}$	ω	$Q_{\acute{\omega}}^{-1}\,(34)\,Q_\omega$
12.34	12.34	$E\,(34)$
34.12	34.12	$(12)\,E$
24.13	23.14	EE
13.24	14.23	EE
14.23	13.24	EE
23.14	24.13	EE

By using the well known Racah factorization lemma, the OPCC can be expressed as the product of an $S_n \supset S_{n-1} \otimes S_1$ outer product isoscalar factor and the coefficient for $S_{n-1,}$ i.e.

$$\left(\begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\lambda] \\ m_1\omega^1 & m_2\omega^2 & m\omega \end{array}\right) = \left\langle\begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\lambda] \\ [\lambda_1-1] & [\lambda_2] & [\lambda-1] \end{array}\right\rangle \times \left(\begin{array}{cc|c} [\lambda_1-1] & [\lambda_2] & [\lambda-1] \\ m_1/m_1^n\left(\omega^1/n\right) & m_2\omega^2 & m/m_n \end{array}\right), \quad (4.12)$$

when the index n is contained in the set ω^1, where m/m_n denotes the Young tableau obtained by removing the square for index n from the Young tableau

m. On the other hand, when the index n is in the ω^2, we have

$$\left(\begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\lambda] \\ m_1\omega^1 & m_2\omega^2 & m\omega \end{array} \right) = \left\langle \begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\lambda] \\ [\lambda_1] & [\lambda_2 - 1] & [\lambda - 1] \end{array} \right\rangle \times \left(\begin{array}{cc|c} [\lambda_1] & [\lambda_2 - 1] & [\lambda - 1] \\ m_1\omega^1 & m_2/m_2^n\left(\omega^2/n\right) & m/m_n \end{array} \right). \quad (4.13)$$

The OPISFs satisfy the unitary conditions

$$\sum_{[\acute{\lambda}_1][\acute{\lambda}_2]\acute{\alpha}} \left\langle \begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\lambda]\,\alpha \\ [\acute{\lambda}_1] & [\acute{\lambda}_2] & [\acute{\lambda}]\,\acute{\alpha} \end{array} \right\rangle \left\langle \begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\bar{\lambda}]\,\bar{\alpha} \\ [\acute{\lambda}_1] & [\acute{\lambda}_2] & [\acute{\lambda}]\,\acute{\alpha} \end{array} \right\rangle = \delta_{[\lambda][\bar{\lambda}]}\delta_{\alpha\bar{\alpha}}, \quad (4.14)$$

$$\sum_{[\lambda]\ \alpha} \left\langle \begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\lambda]\,\alpha \\ [\acute{\lambda}_1] & [\acute{\lambda}_2] & [\acute{\lambda}]\,\acute{\alpha} \end{array} \right\rangle \left\langle \begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\lambda]\,\alpha \\ [\hat{\lambda}_1] & [\hat{\lambda}_2] & [\acute{\lambda}]\,\hat{\alpha} \end{array} \right\rangle = \delta_{[\acute{\lambda}_1][\hat{\lambda}_1]}\delta_{[\acute{\lambda}_2][\hat{\lambda}_2]}\delta_{\acute{\alpha}\hat{\alpha}}. \quad (4.15)$$

We used the same method of evaluating the analytical expressions of the Clebsch-Gordan Coefficients [14] to evaluate analytical expressions of the OPISF. First, we use linear realations among the OPCCs of S_n under the condition (4.3), for the outer product $[\lambda_1] \otimes [\lambda_2] \downarrow [\lambda]$ with smallest n, as long as the IRREPs $[\lambda_1], [\lambda_2]$ and $[\lambda]$ exist. Then, in the multiplicity-free cases, we use (4.12) and (4.13) together with (4.3) to get linear relations of the corresponding OPISFs. The OPISFs are n -indepent. One can replace the OPISF for the specific n by those with general n. Therefore, one can get analytical expressions for the OPISFs from those for specific n. To do this, one has to use analytical expressions of the axial distance in (4.3). Finally, one can use the obtained linear relations together with the unitary condition

$$\sum_{[\acute{\lambda}_1][\acute{\lambda}_2]} \left\langle \begin{array}{cc|c} [\lambda_1] & [\lambda_2] & [\lambda]\,\alpha \\ [\acute{\lambda}_1] & [\acute{\lambda}_2] & [\acute{\lambda}]\,\acute{\alpha} \end{array} \right\rangle^2 = 1, \quad (4.16)$$

to derive all the OPISFs involved in these linear relations up to an overall phase. As examples, OPISFs of $[1] \otimes [n-1]$, $[2] \otimes [n-2]$, $\left[1^2\right] \otimes [n-2]$, $[3] \otimes [n-3]$, $[21] \otimes [n-3]$ and $\left[1^3\right] \otimes [n-3]$ are tabulated in Table (2).

Example: Derivation of OPISFs of $[n-2]\otimes[2]=[n-1,1]$
We need to consider the following expansion

$$\begin{aligned}
\left\| \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \right\rangle = & -a_1\sqrt{\frac{2}{n-1}}\left| \overset{[2]}{\boxed{1\;2}}\quad \overset{[2]}{\boxed{3\;4}} \right\rangle \\
& +a_1\sqrt{\frac{(n-3)^2}{(n-1)(n-2)}}\left| \overset{[2]}{\boxed{1\;3}}\quad \overset{[2]}{\boxed{2\;4}} \right\rangle \\
& +a_1\sqrt{\frac{(n-3)}{(n-1)(n-2)}}\left| \overset{[2]}{\boxed{2\;3}}\quad \overset{[2]}{\boxed{1\;4}} \right\rangle \\
& +a_2\sqrt{\frac{n-2}{n-1}}\left| \overset{[2]}{\boxed{3\;4}}\quad \overset{[2]}{\boxed{1\;2}} \right\rangle \\
& -a_2\sqrt{\frac{1}{(n-1)(n-2)}}\left| \overset{[2]}{\boxed{2\;4}}\quad \overset{[2]}{\boxed{1\;3}} \right\rangle \\
& -a_2\sqrt{\frac{(n-3)}{(n-1)(n-2)}}\left| \overset{[2]}{\boxed{1\;4}}\quad \overset{[2]}{\boxed{2\;3}} \right\rangle, \qquad (4.17)
\end{aligned}$$

where

$$a_1=\left(\begin{array}{cc|c}[n-2]&[2]&[n-1,1]\\ [n-3]&[2]&[n-2,1]\end{array}\right), and \quad a_2=\left(\begin{array}{cc|c}[n-2]&[2]&[n-1,1]\\ [n-2]&[1]&[n-2,1]\end{array}\right). \qquad (4.18)$$

Similarly, we have

$$\begin{aligned}
\left\| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \right\rangle = & -\sqrt{\frac{2(n-3)}{n(n-1)}}\left| \overset{[2]}{\boxed{1\;2}}\quad \overset{[2]}{\boxed{3\;4}} \right\rangle \\
& -\sqrt{\frac{4(n-3)}{n(n-1)(n-2)}}\left| \overset{[2]}{\boxed{1\;3}}\quad \overset{[2]}{\boxed{2\;4}} \right\rangle \\
& -\sqrt{\frac{4}{n(n-1)(n-2)}}\left| \overset{[2]}{\boxed{2\;3}}\quad \overset{[2]}{\boxed{1\;4}} \right\rangle \\
& +\sqrt{\frac{n-2}{n(n-1)}}\left| \overset{[2]}{\boxed{3\;4}}\quad \overset{[2]}{\boxed{1\;2}} \right\rangle \qquad (4.19) \\
& -\sqrt{\frac{n-2}{n(n-1)}}\left| \overset{[2]}{\boxed{2\;4}}\quad \overset{[2]}{\boxed{1\;3}} \right\rangle \\
& -\sqrt{\frac{(n-2)(n-3)}{n(n-1)}}\left| \overset{[2]}{\boxed{1\;4}}\quad \overset{[2]}{\boxed{2\;3}} \right\rangle .
\end{aligned}$$

By acting with $G_3 \to G_{n-1}$ on (4.17), the left hand side of (4.17) becomes

$$
\begin{aligned}
G_{n-1}\left|\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\right\rangle
=&-a_1\sqrt{\frac{2}{n-1}}\left|\overset{[2]}{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}}\right\rangle\\
&-a_2\sqrt{\frac{(n-3)}{(n-1)(n-2)}}\left|\overset{[2]}{\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}}\right\rangle\\
&-a_2\sqrt{\frac{1}{(n-1)(n-2)}}\left|\overset{[2]}{\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}}\right\rangle\\
&+a_2\sqrt{\frac{n-2}{n-1}}\left|\overset{[2]}{\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}\right\rangle\\
&+a_1\sqrt{\frac{(n-3)}{(n-1)(n-2)}}\left|\overset{[2]}{\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}}\right\rangle\\
&+a_1\sqrt{\frac{(n-3)^2}{(n-1)(n-2)}}\left|\overset{[2]}{\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}}\right\rangle,
\end{aligned}
\tag{4.20}
$$

while the right hand side of (4.17) becomes

$$
\begin{aligned}
&\left(-a_1\sqrt{\frac{2}{(n-1)^3}}-\sqrt{\frac{2(n-2)(n-3)}{(n-1)^3}}\right)\left|\overset{[2]}{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}}\right\rangle\\
&+\left(a_1\sqrt{\frac{(n-3)^3}{(n-1)^3(n-2)}}-\sqrt{\frac{4(n-3)}{(n-1)^3}}\right)\left|\overset{[2]}{\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}}\right\rangle\\
&+\left(a_1\sqrt{\frac{(n-3)}{(n-1)^3(n-2)}}-\sqrt{\frac{4}{(n-1)^3}}\right)\left|\overset{[2]}{\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}}\right\rangle\\
&+\left(a_2\sqrt{\frac{(n-2)}{(n-1)^3}}+\sqrt{\frac{(n-2)^2}{(n-1)^3}}\right)\left|\overset{[2]}{\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}\right\rangle\\
&+\left(-a_2\sqrt{\frac{1}{(n-1)^3(n-2)}}+\sqrt{\frac{(n-2)^2}{(n-1)^3}}\right)\left|\overset{[2]}{\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}}\right\rangle\\
&+\left(-a_2\sqrt{\frac{(n-3)}{(n-1)^3(n-2)}}+\sqrt{\frac{(n-2)^2(n-3)}{(n-1)^3}}\right)\left|\overset{[2]}{\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}}\;\overset{[2]}{\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}}\right\rangle.
\end{aligned}
\tag{4.21}
$$

Combining (4.20) and (4.21), we get

$$
a_1=\sqrt{\frac{(n-3)}{(n-2)}}\;\textit{and}\;a_2=\sqrt{\frac{1}{(n-2)}}.
\tag{4.22}
$$

Table (2) Analytical expressions for the OPISFs

Table (2-1) OPISFs of $[1] \otimes [n-1] = [n] + [n-1,1]$

	$[0]\,[n-1]$	$[1]\,[n-2]$
$[n] \downarrow [n-1]$	$\sqrt{\frac{1}{n}}$	$\sqrt{\frac{n-1}{n}}$
$[n-1,1] \downarrow [n-1]$	$\sqrt{\frac{n-1}{n}}$	$-\sqrt{\frac{1}{n}}$

Table (2-2) OPISFs of $[1] \otimes [n-1] = [n-1,1]$

	$[1]\,[n-2]$
$[n-1,1] \downarrow [n-2,1]$	1

Table (2-3) OPISFs of $[2] \otimes [n-2] = [n] + [n-1,1]$

	$[1]\,[n-2]$	$[2]\,[n-3]$
$[n] \downarrow [n-1]$	$\sqrt{\frac{2}{n}}$	$\sqrt{\frac{n-2}{n}}$
$[n-1,1] \downarrow [n-1]$	$\sqrt{\frac{n-2}{n}}$	$-\sqrt{\frac{2}{n}}$

Table (2-4) OPISFs of $[2] \otimes [n-2] = [n-1,1] + [n-2,2]$

	$[1]\,[n-2]$	$[2]\,[n-3]$
$[n-1,1] \downarrow [n-2,1]$	$\sqrt{\frac{1}{n-2}}$	$\sqrt{\frac{n-3}{n-2}}$
$[n-2,2] \downarrow [n-2,1]$	$\sqrt{\frac{n-3}{n-2}}$	$-\sqrt{\frac{1}{n-2}}$

Table (2-5) OPISFs of $[2] \otimes [n-2] = [n-2,2]$

	$[2]\,[n-3]$
$[n-2,2] \downarrow [n-3,2]$	1

Table (2-6) OPISFs of $[1^2] \otimes [n-2] = [n-1,1]$

	$[1]\,[n-2]$
$[n-1,1] \downarrow [n-1]$	1

Table (2-7) OPISFs of $[1^2] \otimes [n-2] = [n-1,1] + [n-2,1,1]$

	$[1]\,[n-2]$	$[1^2]\,[n-3]$
$[n-1,1] \downarrow [n-2,1]$	$-\sqrt{\frac{1}{n}}$	$\sqrt{\frac{n-1}{n}}$
$[n-2,1,1] \downarrow [n-2,1]$	$\sqrt{\frac{n-1}{n}}$	$\sqrt{\frac{1}{n}}$

Table (2-8) OPISFs of $[1^2] \otimes [n-2] = [n-2,1,1]$

	$[1^2]\,[n-3]$
$[n-2,1,1] \downarrow [n-3,1,1]$	1

Table (2-9) OPISFs of $[3] \otimes [n-3] = [n] + [n-1,1]$

	$[2]\ [n-3]$	$[3]\ [n-4]$
$[n] \downarrow [n-1]$	$\sqrt{\frac{3}{n}}$	$\sqrt{\frac{n-3}{n}}$
$[n-1,1] \downarrow [n-1]$	$\sqrt{\frac{n-3}{n}}$	$-\sqrt{\frac{3}{n}}$

Table (2-10) OPISFs of $[3] \otimes [n-3] = [n-1,1] + [n-2,2]$

	$[2]\ [n-3]$	$[3]\ [n-4]$
$[n-1,1] \downarrow [n-2,1]$	$\sqrt{\frac{2}{n-2}}$	$\sqrt{\frac{n-4}{n-2}}$
$[n-2,2] \downarrow [n-2,1]$	$\sqrt{\frac{n-4}{n-2}}$	$-\sqrt{\frac{2}{n-2}}$

Table (2-11) OPISFs of $[3] \otimes [n-3] = [n-2,2] + [n-3,3]$

	$[2]\ [n-3]$	$[3]\ [n-4]$
$[n-2,2] \downarrow [n-3,2]$	$\sqrt{\frac{1}{n-4}}$	$\sqrt{\frac{n-5}{n-4}}$
$[n-3,3] \downarrow [n-3,2]$	$\sqrt{\frac{n-5}{n-4}}$	$-\sqrt{\frac{1}{n-4}}$

Table (2-12) OPISFs of $[3] \otimes [n-3] = [n-3,3]$

	$[3]\ [n-4]$
$[n-3,3] \downarrow [n-4,3]$	1

Table (2-13) OPISFs of $[21] \otimes [n-3] = [n-1,1]$

	$[2]\ [n-3]$
$[n-1,1] \downarrow [n-1]$	1

Table (2-14) OPISFs of $[21] \otimes [n-3] = [n-1,1] + [n-2,2] + [n-2,1,1]$

	$[21]\ [n-4]$	$[2]\ [n-3]$	$[1^2]\ [n-3]$
$[n-1,1] \downarrow [n-2,1]$	$\sqrt{\frac{(n-1)(n-3)}{n(n-2)}}$	$\sqrt{\frac{(n-3)}{2n(n-2)}}$	$\sqrt{\frac{3(n-1)}{2n(n-2)}}$
$[n-2,2] \downarrow [n-2,1]$	$\sqrt{\frac{1}{2(n-2)}}$	$\sqrt{\frac{(n-1)}{4(n-2)}}$	$-\sqrt{\frac{3(n-3)}{4(n-2)}}$
$[n-2,1,1] \downarrow [n-2,1]$	$\sqrt{\frac{3}{2n}}$	$-\sqrt{\frac{3(n-1)}{4n}}$	$-\sqrt{\frac{(n-3)}{4n}}$

Table (2-15) OPISFs of $[21] \otimes [n-3] = [n-2,2] + [n-3,2,1]$

	$[21]\ [n-4]$	$[2]\ [n-3]$
$[n-2,2] \downarrow [n-3,2]$	$\sqrt{\frac{(n-2)}{(n-1)}}$	$\sqrt{\frac{1}{(n-1)}}$
$[n-3,2,1] \downarrow [n-3,2]$	$\sqrt{\frac{1}{(n-1)}}$	$-\sqrt{\frac{(n-2)}{(n-1)}}$

Table (2-16) OPISFs of $[21] \otimes [n-3] = [n-2,1,1] + [n-3,2,1]$

	$[21]\ [n-4]$	$[1^2]\ [n-3]$
$[n-2,1,1] \downarrow [n-3,1,1]$	$\sqrt{\frac{(n-4)}{(n-3)}}$	$\sqrt{\frac{1}{(n-3)}}$
$[n-3,2,1] \downarrow [n-3,1,1]$	$\sqrt{\frac{1}{(n-3)}}$	$-\sqrt{\frac{(n-4)}{(n-3)}}$

Table (2-17) OPISFs of $[21] \otimes [n-3] = [n-3,2,1]$

	$[21]$ $[n-4]$
$[n-3,2,1] \downarrow [n-4,2,1]$	1

Table (2-18) OPISFs of $\left[1^3\right] \otimes [n-3] = [n-2,1,1] + [n-3,1,1,1]$

	$\left[1^3\right]$ $[n-4]$	$\left[1^2\right]$ $[n-3]$
$[n-2,1,1] \downarrow [n-3,1,1]$	$\sqrt{\frac{(n-1)}{n}}$	$\sqrt{\frac{1}{n}}$
$[n-3,1,1,1] \downarrow [n-3,1,1]$	$\sqrt{\frac{1}{n}}$	$-\sqrt{\frac{(n-1)}{n}}$

Table (2-19) OPISFs of $\left[1^3\right] \otimes [n-3] = [n-2,1,1]$

	$\left[1^2\right]$ $[n-3]$
$[n-2,1,1] \downarrow [n-2,1]$	1

Table (2-20) OPISFs of $\left[1^3\right] \otimes [n-3] = [n-3,1,1,1]$

	$\left[1^3\right]$ $[n-4]$
$[n-3,1,1,1] \downarrow [n-4,1,1,1]$	1

References

[1] L. J Somers, J. Math. Phys. **24:** 4 (1983)
[2] Vincenzo Chilla, J. Phys. A **39:** 7657-7668 (2006).
[3] Luke F McAven and Philip H Butler, J. Phys. A **32:** 7509-7522 (1999).
[4] L F McAven, P H Butler, and A M Hamel, J. Phys. A **31:** 8363-8372 (1998).
[5] J. Q. Chen, **Group Representation Theory for Physicists** (World Scientific, Singapore, 1989).
[6] Q. Zhang and Xiangzhu LI, J. Phys. A **20:** 6185-6196(1987).
[7] Xiangzhu LI, J. Phys. A **22:** 2299-2307 (1989).
[8] Xiangzhu LI, J. Phys. A **28:** 1361-1370 (1995).
[9] Zhang Qianer and LI Xiangzhu, Int. J. Q. Chem XXXVI: 201-211 (1989)
[10] Xiangzhu LI and Josef Paldus, J. Math. Chem. **4:** 295-353 (1990).
[11] F. Pan and J. Q. Chen, J. Phys. A **26:** 4299 (1993).
[12] F. Pan, J. Phys. A **28:** 3139 (1995).
[13] F. Pan and L. Dai, J. Phys. A **29:** 5079-5091(1996).
[14] L. Dai, F. Pan and J. P. Draayer J. Math. Phys. **43:** 12 (2002).
[15] F. Pan, Commun. Theor. Phys. **31:** 113 (1999).
[16] M. Hamermesh, **Group Theory and Its Applications to Physical Problems,** Addison-Wesley, Reading, MA, (1962).
[17] V. Vanagas, **Algebraic Method in Nuclear Theory,** Mintis, Vilnius, (1972).
[18] I. G. Kaplan, **Symmetry of Many Electron Systems,** Academic Publishing Company, New York, (1975).