Applications of the Collective Model to Some Axially Symmetric Deformed Even-Even Nuclei

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Abstract
The rotational energies of deformed even-even nuclei, which have axes of symmetry, are investigated. A new simple formula, which determines the energy of the rotational state of a deformed even-even nucleus in terms of its total angular momentum I and its reciprocal moment of inertia $\frac{\hbar^2}{2\Im}$, is given. The new formula is fitted for the even-even isotopes of the hafnium: ^{168}Hf, ^{170}Hf, ^{172}Hf and ^{174}Hf, uranium: ^{232}U, ^{234}U, and ^{236}U, and plutunium: ^{238}Pu, ^{240}Pu, and ^{242}Pu nuclei. The required reciprocal values of the cranking moments of inertia of these isotopes are calculated by using the concept of the single-particle Schroedinger fluid and are in excellent agreement with the corresponding experimental values. Moreover, the calculated values of the rotational energies of the mentioned ten isotopes are in good agreement with the corresponding experimental values and are better than those obtained by using previous formulas. Furthermore, the ratio of the excitation energies of the first 4^+ and the first 2^+ excited states, $R_{4/2}$, is also calculated for these nuclei. Finally, new simple formulas for calculating the energies of the β-band and the γ-band vibrational states are given and used for these nuclei.

Key words: Nuclear deformation, collective motions, moments of inertia, rotational states of even-even nuclei, rotational-vibrational model.

1. Introduction

As protons or neutrons are added to a closed−shell nucleus, the nucleus may shift to oblate, prolate or triaxial shape. In these instances, the spherical shell model is inadequate and collective models, such as that proposed by Bohr and Mottelson [1], must instead be brought to bear. Again, for nuclei that are in regions of strong deformation, these models have had a wide array of successes.

It is well known that rotational bands in deformed nuclei and their electromagnetic transitions are fundamental manifistations of nuclear collective modes. The traditional approach to study them is via the geometrical model of Bohr and Mottelson [1]. It is also well known that nuclei are not rigid. This can be seen from the fact that as the spin momentum J becomes higher, the nuclear moment of inertia will generally increase as well. This fact is known as stretching.

Over the years some phenomenological models have been introduced to characterize the properties of rotational bands for nuclei. For even-even nuclei with large values of the mass number A, the rotational energy levels are well known experimentally, for even values of the total spin momentum I between 2 and 20.

Several attempts have been made to fit the nuclear energy levels as functions of the total spin momentum I and the nuclear moment of inertia $\Im$ [2-7].

Ahmadov et al [8] introduced a simple formula for the energy level density of even–even nuclei in the region of rare-earth and actinide elements.

A unified approach to describe the properties of spherical, transition, and deformed even-even nuclei is produced by Mitroshin et al [9].

One of the successes of the collective models is the prediction and subsequent observation of nuclear levels that arise from a combination of nuclear vibration and rotation in deformed nuclei. These vibrational states can be characterized in terms of their symmetry axes. A state that is formed from vibrations about a nucleus, for which long axis is termed a β vibrational band while a γ vibrational band is formed when the short sides of the nucleus pulsate, is deviating from spherical symmetry along the long axis [10].

In the present paper we have introduced a new simple formula for calculating the rotational energies of a deformed nucleus as function of the total spin quantum number I and the nuclear moment of inertia $\Im$. This formula depends on three parameters, one of which is the reciprocal moment of inertia. In the numerical calculations we have firstly calculated the cranking moments of inertia of the ten even-even isotopes: ^{168}Hf, ^{170}Hf, ^{172}Hf, and ^{174}Hf; ^{232}U, ^{234}U, and ^{236}U; and ^{238}Pu, ^{240}Pu, and ^{242}Pu, by using the concept of the single-particle Schroedinger fluid [11]. Accordingly, one of the three parameters of our new formula for calculating the rotational energy levels has been fixed. The other two parameters are then calculated by suitable fit with the values of some experimental levels. Furthermore, we have calculated the rotational energy levels of the ten mentioned isotopes by using our new formula and other five well-known formulas. Comparison between the obtained results of the rotational energies for the mentioned ten isotopes by using the new formula, the other five formulas and the corresponding experimental values is also given.
Furthermore, the ratio of the excitation energies of the first 4^+ and the first 2^+ excited states, $R_{4/2}$, is also calculated for these nuclei. Finally, new simple formulas for calculating the energies of the β-band and the γ-band vibrational states are given and used for these nuclei.

2. Rotational Energies

The existence of collective energy level bands of rotational and vibrational types can now be easily identified from the data of nuclear spectra for many deformed light even-even nuclei. Also, it is well known that deformed nuclei have rotational energy spectra due to their collective motions. These energy spectra are generally defined by the total angular momentum, I, the parity, π, and the quantum number K, which is the projection of the angular momentum on the intrinsic coordinate axis of the nucleus. Here, K, which is the conserved quantum number for the nuclei having axial symmetry, could take values $K = I, I-1, \dots, -I$. Indeed, K, which is defined as the angular momentum of the nuclei, depending on the attached coordinate system, has a definite and constant value for a specified intrinsic state of any deformed nuclei. The intrinsic energy E_K of slowly rotating nuclei depends on the quantum number K and it forms the base of the rotational energy band, and the energy eigenvalues of such a band could be defined as [12]

$$E(I,K) = E_{rot}(I) + E_{K}. \qquad (2.1)$$

The symmetric top is a special case for which the rotational Hamiltonian is particularly simple and explicit formulas can be obtained for the rotational energies [2]

$$E(I,K)=\frac{\hbar^2}{2\Im}\left[I(I+1)-K^2\right]+\frac{\hbar^2}{2\Im_Z}(K-\Omega)^2. \tag{2.2}$$

In the limit of a perfect axial rotor, the rotational angular momentum $\boldsymbol{R}$ is perpendicular to the z-axis, and Ω tends to K. In that case, the second term in equation (2.2) vanishes, and we obtain the rotational energy for an axially symmetric nucleus

$$E(I,K)=\frac{\hbar^2}{2\Im}\left[I(I+1)-K^2\right]. \tag{2.3}$$

Nuclei are many-body systems of nucleons exhibiting a rich variety of nuclear structure properties associated with single-particle and collective motions. A central theme in nuclear structure has been in evaluating these seemingly contrasting facts of nuclear dynamics [1]. Collective nuclear motion involves two basic degrees of freedom, rotations and vibrations, with the latter being described as elementary excitation modes that are superimposed on the equilibrium nuclear shape. The lowest order oscillations in deformed nuclei that affect shape changes are quadrupole in nature resulting in the two well-known types of vibrations: β with no projection of the angular momentum on the symmetry axis $(K=0)$ and in the low-spin region. The phenomenological analysis of spectra of well deformed even-even nuclei has gone on for more than decades since Bohr and Mottelson described the ground- state rotational bands with a rigid rotor expression of the form [2]

$$\mathrm{E}(I)=\frac{\hbar^2}{2\Im}\left[I(I+1)\right]=\mathrm{A}I(I+1). \tag{2.4}$$

For higher values of I, deviations occur for nuclei. In order to describe the deviation from the ideal rotational spectrum, the energy $E(I)$ may be expanded as an infinite power series of the quantity $I(I+1)$ in the form [1,3]

$$\mathrm{E}(I)=\sum_{\mathrm{j}=1}\mathrm{A_j}[I(I+1)]^{\mathrm{j}}. \tag{2.5}$$

Up to now a variety of empirical formulas have been developed to correct the systematic deviation between Eq.(2.4) and experiment. Among these formulas two parameter ones have been used more generally for their less parameters. A brief summary is as follows [13]:

(i) AB-formula

In principle, the rotational energy may be expanded as an infinite power series of $I(I+1)$. The first order approximation is of the form

$$\mathrm{E}(I)=\mathrm{A}I(I+1)-\mathrm{B}I^2(I+1)^2, \tag{2.6}$$

which is known as the AB-formula [13].

(ii) Warke-Khadikikar formula

A variant of the AB formula known as the Warke-Khadikikar formula [4] is obtained as follows:

If the magnitude of the second term in Eq. (2.6) is small enough compared with the first one, one can rewrite the equation as

$$\mathrm{E}(I) = \mathrm{A}I(I+1)\left[1 - \frac{\mathrm{B}}{\mathrm{A}}\, I(I+1)\right] \approx \frac{\mathrm{A}I(I+1)}{1+\frac{\mathrm{B}}{\mathrm{A}} I(I+1)}. \tag{2.7}$$

The difference between Eqs. (2.6) and (2.7) is that the latter is convergent as I increases. Using such a simple formula, Bonatsos [14] has accounted for the gradual increase of the moment of inertia with angular momentum below band crossing in the actinide and rare-earth regions.

(iii) Harris formula

Harris [5] proposed a two-parameter formula for the nuclear rotational spectrum in terms of the rotational angular frequency ω, in the form

$$E = \alpha\omega^2 + \beta\omega^4, \tag{2.8}$$

(iv) Variable moment of inertia formula

Taking the variable moment of inertia of the deformed nucleus into consideration, the following formula for the rotational energy has been obtained [6]

$$E(I) = \frac{\hbar^2}{2\Im} I(I+1) + \frac{1}{2}\mathrm{C}(\Im - \Im_0), \tag{2.9}$$

where $\Im_0$ is determined from the variational condition $\frac{\partial \mathrm{E(I)}}{\partial \Im} = 0$. It has been proved also that the variable moment of inertia formula is equivalent to the $\alpha\beta$ −formula of Harris [5].

(v) The ab-formula

Abundant information about the high spin levels of gound rotational bands (GRB's) in actinide nuclei have been obtained from Coulomb excitation experiments with very heavy ion beams. Because the moments of inertia of actinide nuclei are about twice as big as those of the rare-earth nuclei, the two quasiparticles bands do not compete with the GRB up on much higher spins than those in rare-earth nuclei. Holmberg and Lipas [7], making use of the available data, deduced a simple formula from the nuclear hydrodynamical model to fit data of the nuclear rotational spectra, called the ab formula and is given by

$$\mathrm{E}(I) = \mathrm{a}\left[\sqrt{1 + \mathrm{b}I(I+1)} - 1\right]. \tag{2.10}$$

It has been deduced by investigating the curves of the moment of inertia $\Im$ against the rotational energy E.

3. The Rotation-Vibration model (RVM)

The systematic solutions for an axially symmetric deformed nucleus with β- and γ-vibration were obtained by Faessler and Greiner [15-21] and are given by

(a) The ground state band

$$E = I(I+1)\frac{1}{2}\varepsilon\left(1+3\frac{\varepsilon}{E_\gamma}+\frac{3}{2}\frac{\varepsilon}{E_\beta}+2\frac{\varepsilon}{E_\gamma}\frac{\varepsilon}{E_\gamma-2\varepsilon}\right)$$
$$-I^2(I+1)^2\varepsilon\left(\frac{1}{2}\frac{\varepsilon}{E_\gamma}\frac{\varepsilon}{E_\gamma-2\varepsilon}+\frac{3}{2}\left(\frac{\varepsilon}{E_\beta}\right)^2\right) \quad (3.1)$$

(b) The γ-band

$$E_{\gamma-band} = E_\gamma$$
$$+I(I+1)\frac{1}{2}\varepsilon\left(1+6\frac{\varepsilon}{E_\gamma}+\frac{3}{2}\frac{\varepsilon}{E_\beta}-2\frac{\varepsilon}{E_\gamma}\frac{\varepsilon}{E_\gamma-2\varepsilon}\delta_{I:2,4,\ 6,\dots,}\ -\ 27\frac{\varepsilon}{E_\gamma-E_\beta-2\varepsilon}\frac{\varepsilon}{E_\gamma}\frac{\varepsilon}{E_\beta}\right)+$$
$$I^2(I+1)^2\varepsilon\left(\frac{1}{2}\frac{\varepsilon}{E_\gamma}\frac{\varepsilon}{E_\gamma-2\varepsilon}+\frac{27}{4}\frac{\varepsilon}{E_\gamma-E_\beta-2\varepsilon}\frac{\varepsilon}{E_\gamma}\frac{\varepsilon}{E_\beta}\delta_{I:0,2,4,\ 6,\dots,}\right)-2\varepsilon(1+6\frac{\varepsilon}{E_\gamma}+\frac{3}{2}\frac{\varepsilon}{E_\beta}) \quad (3.2)$$

(c) The β-band

$$E_{\beta-band} = E_\beta + I(I+1)\frac{1}{2}\varepsilon\left(1+3\frac{\varepsilon}{E_\gamma}+\frac{27}{2}\frac{\varepsilon}{E_\beta}-\frac{81}{2}\frac{\varepsilon}{E_\gamma}\frac{\varepsilon}{E_\beta}\frac{\varepsilon}{E_\gamma-E_\beta+2\varepsilon}\ +12\,(\frac{\varepsilon}{E_\beta})^2\right)+$$
$$I^2(I+1)^2\varepsilon\left(\frac{27}{2}\frac{\varepsilon}{E_\gamma}\frac{\varepsilon}{E_\beta}\frac{\varepsilon}{E_\gamma-E_\beta+2\varepsilon}+\frac{3}{2}\,(\frac{\varepsilon}{E_\beta})^2\right)+6(\frac{\varepsilon}{E_\beta})^2 \quad (3.3)$$

4. The New Rotational and Vibrational Formulas

Other formulas for calculating the rotational energies of deformed even-even nuclei can be found in [22-24].

By analysing the well known experimental rotational energy levels of the axially deformed even-even nuclei in the high mass region we have derived a new formula for the rotational energy levels, that depends upon the total spin momentum I and the nuclear moment of inertia $\Im$ in the following simple form

$$E(I) = \frac{AI(I+1)}{\left[1+\frac{DI(I+1)}{1-CI(I+1)}\right]}. \quad (4.1)$$

Here, A is the reciprocal moment of inertia of the nucleus, $A = \frac{\hbar^2}{2\Im}$. The value of A has been determined for all the considered isotopes by using the concept of the single-particle Schroedinger fluid [11].

Accordingly, our formula contains two parameters beside the nuclear moment of inertia. In our fitting we determined C and D by inserting two values of the experimental rotational energies for middle values of I. In our calculations we considered $\mathrm{I} = 10$ and $\mathrm{I} = 12$ to determine C and D.

When $\mathrm{D} = \mathrm{C}$ formula (4.1) gives the following simple relation

$$E(I) = AI(I+1) - BI^2(I+1)^2, \quad (4.2)$$

where $B = AC$. This special case coincides with the AB-formula, Eq. (2.6).

Accordingly, our new formula modifies the AB- formula by the correction factor:

$$\frac{1}{\{1+(D-C)[I(I+1)]\}}.$$

Furthermore, by analysing the well known experimental β-band energy levels of the axially deformed even-even nuclei in the high mass region we have derived a new formula for the β-band energy levels, that depends upon the total spin momentum I, head band of β and head band of γ and the nuclear moment of inertia $\Im$ in the following simple form,

$$E_{\beta-band} = -\frac{\hbar^2}{\Im}\frac{1}{E_\gamma} I^2(I+1)^2 + \frac{\hbar^2}{2\Im} I(I+1) + E_\beta. \qquad (4.3)$$

Also, by analysing the well known experimental γ-band energy levels of the axially deformed even-even nuclei in the high mass region we have derived a new formula for the γ-band energy levels, that depends upon the total spin momentum I, head band of γ, the number of neutrons N and the nuclear moment of inertia $\Im$ in the following simple form

$$E_{\gamma-band} = \frac{\hbar^2}{2\Im}\frac{1}{N} I^2(I+1)^2 + \frac{1}{2}\frac{\hbar^2}{2\Im} I(I+1) + E_\gamma \ . \qquad (4.4)$$

5. The $R_{4/2}$ Classification

If the rotational energies of the deformed even-even nuclei are grouped into classes, within each class the ratio:

$$R_{4/2} = \frac{E(4_1^+)}{E(2_1^+)}, \qquad (5.1)$$

of excitation energies of the first 4^+ and the first 2^+ excited states, must lie in a fixed interval [24]. The use of the parameter (5.1) as an indicator of collective dynamics is justified both empirically and by theoretical arguments. Some of these arguments are [24]:
(i) For all nuclei with $38 \leq Z \leq 82$ and with $2.05 \leq R_{4/2} \leq 3.15$ the data fall on a straight line. This suggests that nuclei in this wide range of Z-values behave like anharmonic vibrators with nearly constant anharmonicity. It was found also that a linear relation between $E(4^+)$ and $E(2^+)$ holds for pre-collective nuclei with $R_{4/2} < 2$.
(ii) Theoretical calculations based on the Interacting Boson Model-1 (IBM-1) support the conclusion that $R_{4/2}$ is an appropriate measure for collectivity in nuclei. The IBM calculation of energy levels yields values of $R_{4/2} = 2.00, 3.33,$ and 2.5 for the dynamical symmetries $U(5), SU(3)$ and $O(6)$, respectively.
(iii) The calculation of energy levels for the IBM Hamiltonian yields $R_{4/2} = 2.20$ and 2.91 for the critical symmetries $E(5)$ and $X(5)$, respectively.

6. Results and Conclusions

In Table-1 we present the values of the reciprocal moments of inertia (A) for the ten isotopes: ^{168}Hf, ^{170}Hf, ^{172}Hf, and ^{174}Hf; ^{232}U, ^{234}U, and ^{236}U; and ^{238}Pu, ^{240}Pu, and ^{242}Pu by applying the concept of the single-particle Schroedinger fluid [11] together with the corresponding values of the deformation parameter β, which reproduces the best value of A, The corresponding experimental values of A are also given in Table-1. Moreover, the values of the other parameters C and D are also given in this table. Furthermore, the values of the ratio $R_{4/2}$ are also given in Table-1.

In the numerical calculations of the rotational energies of the ten axially deformed even-even isotopes: ^{168}Hf, ^{170}Hf, ^{172}Hf, and ^{174}Hf; ^{232}U, ^{234}U, and ^{236}U; and ^{238}Pu, ^{240}Pu, and ^{242}Pu we have used all the five mentioned above formulas together with the new formula. Among the five formulas the results obtained by using the ab-formula are to some extent better than those of the other four formulas. Accordingly, we present only in Table-2 the calculated values of the rotational energies of the mentioned ten isotopes, for even values of the total angular momentum I in the interval from 2 to 20, by using the ab-formula and the new formula together with the available experimental values, as functions of the total spin I. The experimental values are taken from [25-31].

Table-1 The reciprocal moments of inertia (*A*) of the ten isotopes together with the corresponding values of the deformation parameter β which reproduces the best value of *A*. The corresponding experimental values of *A* are also given. Moreover, the values of the other parameters *C* and *D* are also given. Furthermore, the values of the ratio $R_{4/2}$ are also given.

Nucleus	β	$A_{S.f.}$	A_{exp}	C	D	$R_{4/2}$
^{168}Hf	0.32 -0.27	20.41 20.48	20.4	-7.847812E-02	1.083755E-02	3.21
^{170}Hf	0.15 -0.17	17.00 17.08	17.1	3.735062E-02	5.949669E-04	3.23
^{172}Hf	0.26 -0.34	15.86 15.92	15.85	-3.919555E-02	3.130735E-03	3.27
^{174}Hf	0.25 -0.32	15.01 15.17	15	5.405125E-02	-6.231677E-05	3.28
^{232}U	0.19 -0.20	8.27 8.32	8.28	-3.973872E-01	2.616873E-02	3.30
^{234}U	0.19 -0.20	7.30 7.25	7.29	-4.410045E-02	1.890931E-03	3.30
^{236}U	0.20 -0.21	7.56 7.49	7.57	-1.718663E-02	8.948615E-04	3.31
^{238}Pu	0.18 -0.19	7.38 7.42	7.37	-2.104008E-01	5.816812E-03	3.31
^{240}Pu	0.32 -0.30	7.15 7.08	7.16	4.634872E-02	2.808613E-05	3.31
^{242}Pu	0.34 -0.31	7.43 -7.34	7.42	2.662025E-02	1.887374E-04	3.31

It is seen from Table-2 that the calculated values of the rotational energies of the ten isotopes by using the new formula are in better agreement than the corresponding ones by using the ab-formula.

In Table-3 we present the β-band energies of ^{168}Hf, ^{170}Hf, ^{172}Hf, and ^{174}Hf; ^{232}U, ^{234}U, and ^{236}U; and ^{238}Pu, ^{240}Pu, and ^{242}Pu as functions of the total spin I by using the new formula. The experimental values are taken from [25-31].

In Table-4 we present the γ-band energies of ^{168}Hf, ^{170}Hf, ^{172}Hf, and ^{174}Hf; ^{232}U, ^{234}U, and ^{236}U; and ^{238}Pu, ^{240}Pu, and ^{242}Pu as functions of the total spin I by using the new formula. The experimental values are taken from [25-31].

It is seen from Table-3 that the calculated values of the β-band energies of the ten isotopes by using the new formula are in good agreement with the experiment values.

Also, it is seen from Table-4 that the calculated values of the γ-band energies of the ten isotopes by using the new formula are in good agreement with the experiment values.

In Figs. 1-3 we present the variations of the rotational and vibrational energies of the ten isotopes with respect to the total angular momentum I by using the new formulas and the corresponding experimental values.

It is seen from the ten figures that the obtained results by using the new formulas are in very good agreement with the corresponding experimental values. This show that the new formulas may be used successfully in the calculations of the rotational and vibrational energies of other axially deformed even-even nuclei.

Moreover, we notice from Table-1 that all the considered isotopes have values of the ratio $R_{4/2}$ between 3.21 and 3.31. This shows that the corresponding energy levels have $SU(3)$ dynamical symmetry.

Table-2 Rotational energies of ^{168}Hf, ^{170}Hf, ^{172}Hf, and ^{174}Hf; ^{232}U, ^{234}U, and ^{236}U; and ^{238}Pu, ^{240}Pu, and ^{242}Pu as functions of the total spin *I* by using the ab-formula and the new formula. The experimental values are taken from [25-31].

	Case	E(I) in KeV									
		I = 2	I = 4	I = 6	I = 8	I = 10	I = 12	I = 14	I = 16	I = 18	I = 20
^{168}Hf	ab	120.38	385.54	766.47	1233.35	1761.93	2334.54	2938.8	3566.27	…	…
	new	121.85	385.29	755.84	1213.50	1735.70	2305.80	2938.78	3599.30	4260.34	4995.71
	exp	123.95	385.54	756.8	1213.5	1735.7	2305.8	2938.8	3622.9	…	…
^{170}Hf	ab	99.98	322.54	646.94	1050.43	1513.11	2019.55	2558.4	3121.48	…	…
	new	99.97	321.76	641.94	1043.30	1505.50	2016.40	2567.50	3158.70	3769.16	4422.23
	exp	100.8	321.99	642.9	1043.3	1505.5	2016.4	2567.2	3151.6	3766.5	4421
^{172}Hf	ab	94.79	309.41	630.24	1040.43	1523.21	2063.76	2649.88	3271.91	…	…
	new	94.93	309.11	627.77	1037.30	1521.07	2064.44	2656.70	3279.71	3921.30	4672.62
	exp	95.24	309.26	628.14	1037.3	1521.07	2064.44	2654.01	3277.17	3919.4	4575.9
^{174}Hf	ab	90.99	297.75	608.53	1008.44	1482.19	2015.82	2597.5	3217.62	…	…
	new	90.94	297.24	607.98	1009.60	1485.90	2020.50	2597.68	3209.45	3855.80	4548.6
	exp	90.99	297.38	608.26	1009.6	1485.9	2020.5	2597.5	3208.9	3857.3	4550.8
^{232}U	ab	47.4	156.34	323.13	542.65	809.03	1116.28	1458.65	1830.96	2228.65	2647.82
	new	47.48	156.38	322.56	541.00	805.80	1111.50	1453.40	1828.04	2232.71	2671.84
	exp	47.57	156.56	322.6	541	805.8	1111.5	1453.7	1828.1	2231.5	2659.7
^{234}U	ab	43.18	142.65	295.51	497.65	744.3	1030.38	1350.95	1701.41	2077.5	2475.8
	new	43.49	143.32	296.06	497.04	741.20	1023.80	1340.80	1689.16	2066.60	2470.9
	exp	43.5	143.35	296.07	497.04	741.2	1023.8	1340.8	1687.8	2063	2469.2
^{236}U	ab	45.15	149.48	309.71	522.52	783.08	1087.46	1427.69	1802.1	2205.36	2633.65
	new	45.20	149.47	309.75	522.24	782.30	1085.30	1426.99	1801.60	2205.30	2630.89
	exp	45.24	149.48	309.78	522.24	782.3	1085.3	1426.3	1800.9	2203.9	2631.7
^{238}Pu	ab	43.97	145.74	303.43	514.25	774.78	1081.18	1429.41	1815.43	2235.37	…
	new	44.08	145.94	303.39	513.40	772.80	1078.50	1427.80	1817.86	2250.01	2720.37
	exp	44.08	145.98	303.4	513.4	772.8	1078.5	1427.2	1816.2	2240.5	…
^{240}Pu	ab	42.77	141.64	294.45	498.09	748.75	1042.26	1374.35	…	…	…
	new	42.83	141.71	294.42	497.52	747.80	1041.80	1376.17	1747.76	2154.64	2592.01
	exp	42.82	141.69	294.32	497.52	747.8	1041.8	1375.6	1745.7	2151.1	2590.2
^{242}Pu	ab	44.41	147.12	306.07	518.22	779.87	1086.9	1435.03	1820.06	2237.95	2684.96
	new	44.52	147.38	306.36	518.10	778.70	1084.00	1430.91	1815.69	2236.09	2687.45
	exp	44.54	147.3	306.4	518.1	778.7	1084	1431.3	1816.3	2235.6	2686.6

Table-3 β-band energies of ^{168}Hf, ^{170}Hf, ^{172}Hf, and ^{174}Hf; ^{232}U, ^{234}U, and ^{236}U; and ^{238}Pu, ^{240}Pu, and ^{242}Pu as functions of the total spin *I* by using the new formula. The experimental values are taken from [25-31].

nucleus		0^+	2^+	4^+	6^+	E_β	E_γ	A
$^{168}_{72}Hf$	exp	942.09	1058.6	1284.66	-------	942.09	833.3	21.3 gs
	cal	942.09	1062.7	1330.5	1712.5			20.4 bb
$^{170}_{72}Hf$	exp	879.6	987.0	1156.6	------	879.6	923.3	17.1 gs
	cal	879.6	985.6	1222.1	1563.0			17.9 bb
$^{172}_{72}Hf$	exp	871.3	952.43	1129.52	------	871.3	1044.5	15.8
	cal	871.3	965.01	1175.2	1481.5			The same
$^{174}_{72}Hf$	exp	827.8	900.24	1062.17	1307.4	827.8	1196	15 gs
	cal	827.8	897.3	1053.97	1284.7			11.7 bb
$^{232}_{92}U$	exp	691.3	734.57	833.07	948.9	691.3	850.95	7.92
	cal	691.3	738.1	842.3	991.1			The same
$^{234}_{92}U$	exp	809.8	851.74	947.64	1096.1	809.8	912.3	7.2
	cal	809.8	852.4	947.49	1084.4			The same
$^{236}_{92}U$	exp	919.14	960.3	1050.85	-------	919.14	942.9	7.5
	cal	919.14	963.57	1062.78	1206.1			The same
$^{238}_{94}Pu$	exp	941.46	983.09	----	------	941.46	1013	7.37
	cal	941.46	985.16	1088.03	1225.3			The same
$^{240}_{94}Pu$	exp	860.7	900.32	992.4	1138.3	860.7	1121.7	7.16
	cal	860.7	903.2	998.7	1138.9			The same
$^{242}_{94}Pu$	exp	956	992.5	----	------	956	1087.1	7.43 gs
	cal	956	994.57	1081.22	1207.9			6.5 bb

Table-4 γ-band energies of ^{168}Hf, ^{170}Hf, ^{172}Hf, and ^{174}Hf; ^{232}U, ^{234}U, and ^{236}U; and ^{238}Pu, ^{240}Pu, and ^{242}Pu as functions of the total spin *I* by using the new formula. The experimental values are taken from [25-31].

nucleus		2^+	3^+	4^+	5^+	E_β	E_γ	A
$^{168}_{72}Hf$	exp	875.94	1030.93	1160.71	1386.38	942.09	833.3	21.3
	cal	905.19	993.05	1135.05	1352.49			
$^{170}_{72}Hf$	exp	961.30	1087.5	1227.3	-------	879.6	923.3	16.7
	cal	979.5	1048.04	1158.46	1327.17			
$^{172}_{72}Hf$	exp	1075.29	1180.87	1304.66	1462.88	871.3	1044.5	15.8
	cal	1097.6	1162.05	1265.7	1423.7			
$^{174}_{72}Hf$	exp	1226.77	1336.48	1448.85	1658.41	827.8	1196	15
	cal	1246.29	1307.18	1404.8	1553.4			
$^{232}_{92}U$	exp	866.79	911.49	970.71	------	691.3	850.95	7.92
	cal	876.75	906.62	952.78	1020.66			
$^{234}_{92}U$	exp	926.72	968.42	1023.77	1090.89	809.8	912.3	7.2
	cal	935.7	965.66	1012.54	1083.83			
$^{236}_{92}U$	exp	957.90	1001.5	1058.8	1127.38	919.14	942.9	7.5
	cal	967.28	995.4	1038.73	1102.28			
$^{238}_{94}Pu$	exp	1028.54	1069.94	1125.76	------	941.46	1013	7.37
	cal	1036.95	1064.59	1107.17	1169.61			
$^{240}_{94}Pu$	exp	1136.97	1177.63	1232.46	------	860.7	1121.7	7.16
	cal	1144.9	1171.7	1212.9	1273.24			
$^{242}_{94}Pu$	exp	1102	-----	------	-------	956	1087.1	7.43
	cal	1111.19	1138.9	1181.48	1243.73			

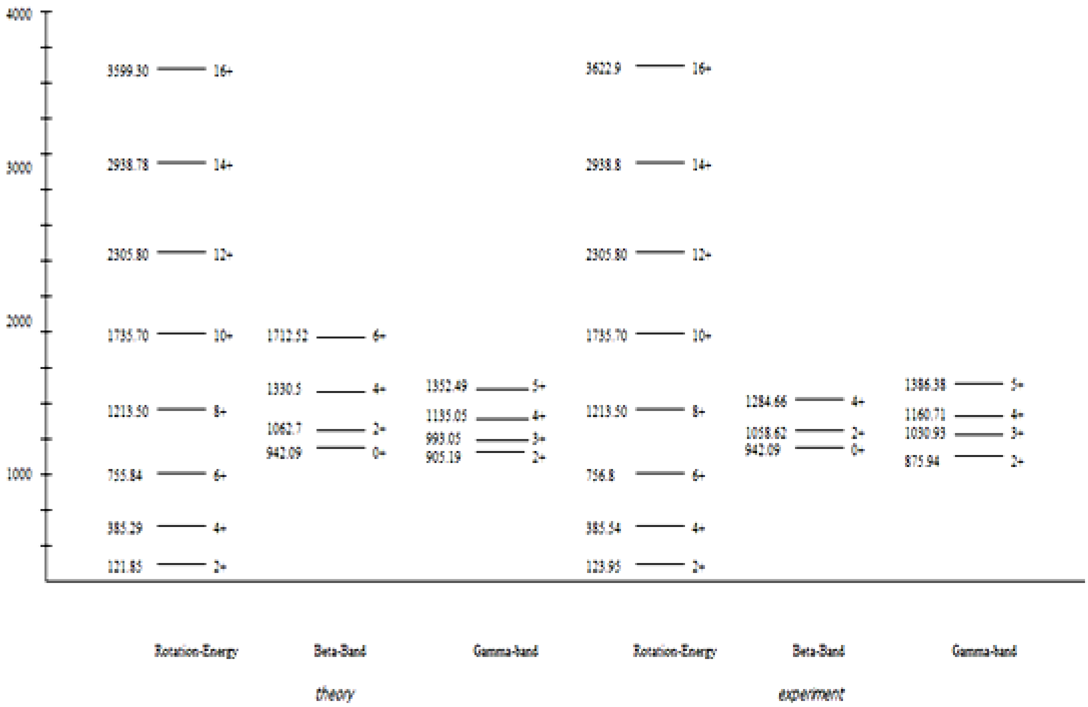

Fig. 1-a (^{168}Hf)

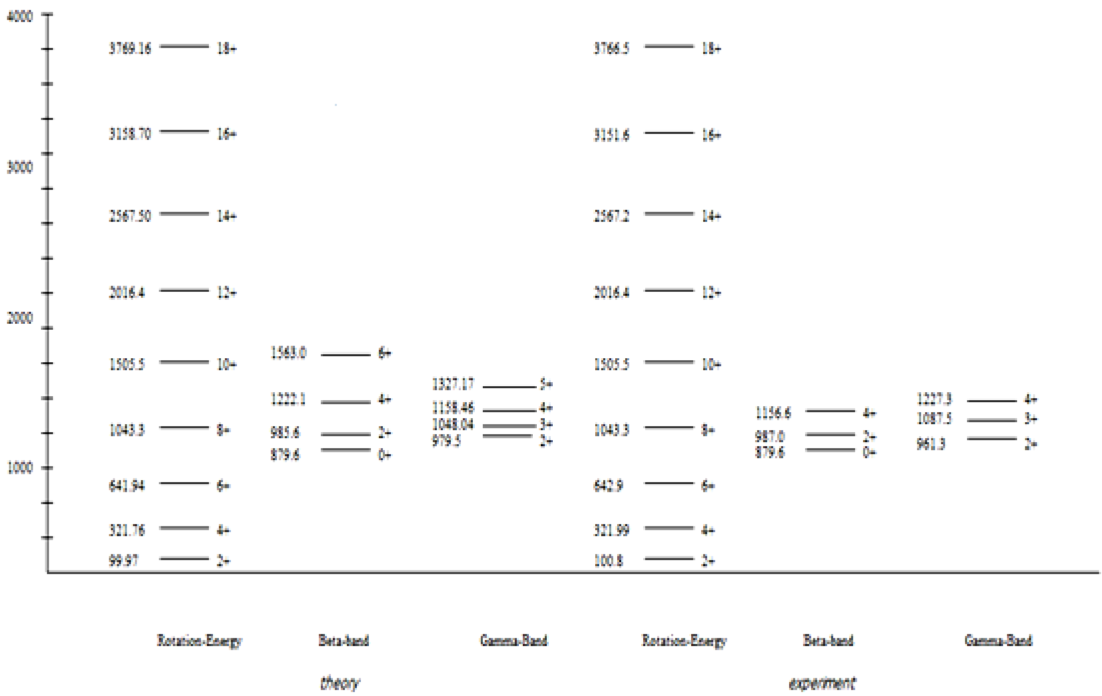

Fig. 1-b (^{170}Hf)

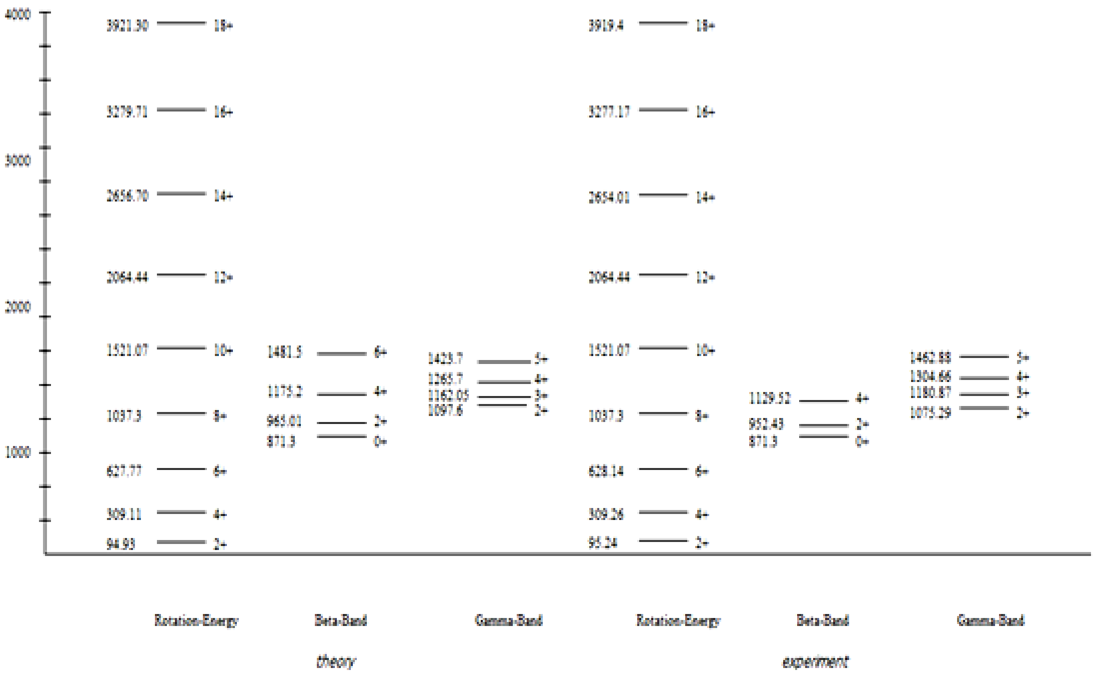

Fig. 1-c (^{172}Hf)

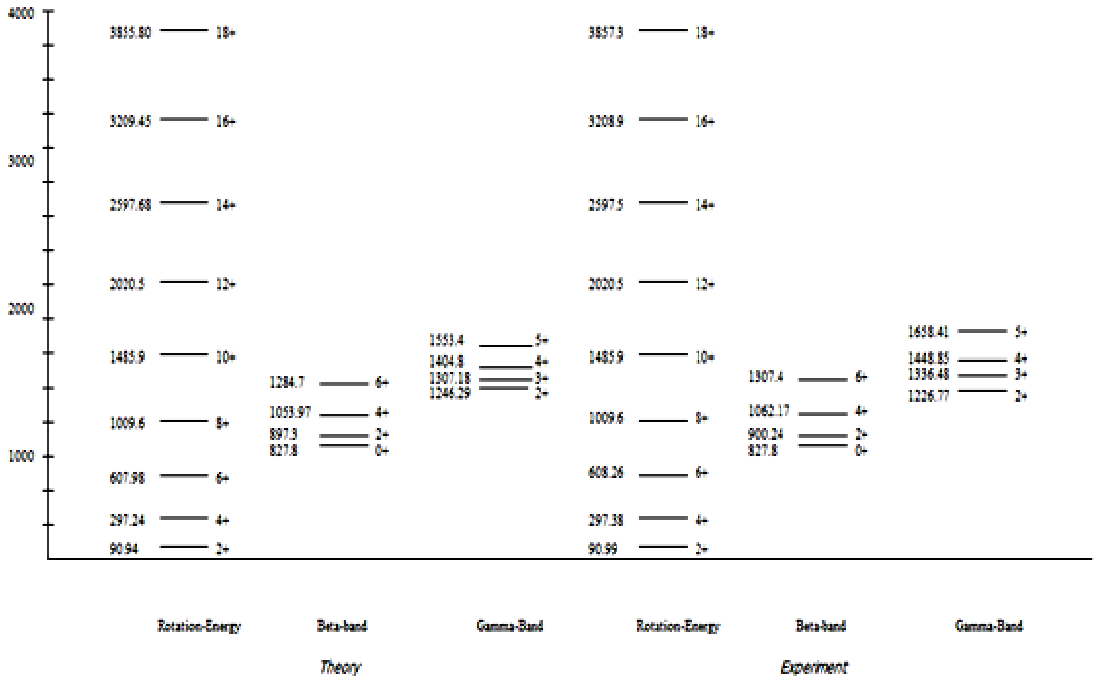

Fig.1-d (^{174}Hf)

Figs. 1-a, 1-b, 1-c and 1-d. Variations of the rotational and vibrational energy eigenvalues of the hafnium isotopes: ^{168}Hf, ^{170}Hf, ^{172}Hf and ^{174}Hf with respect to the total angular momentum I.

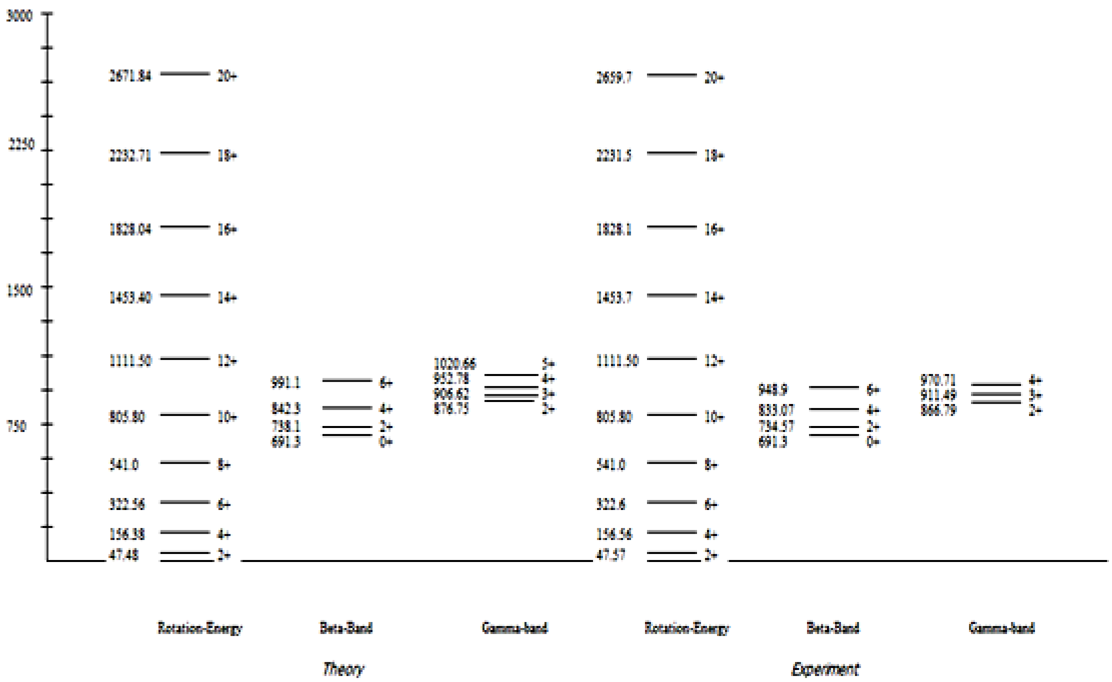

Fig.2-a (^{232}U)

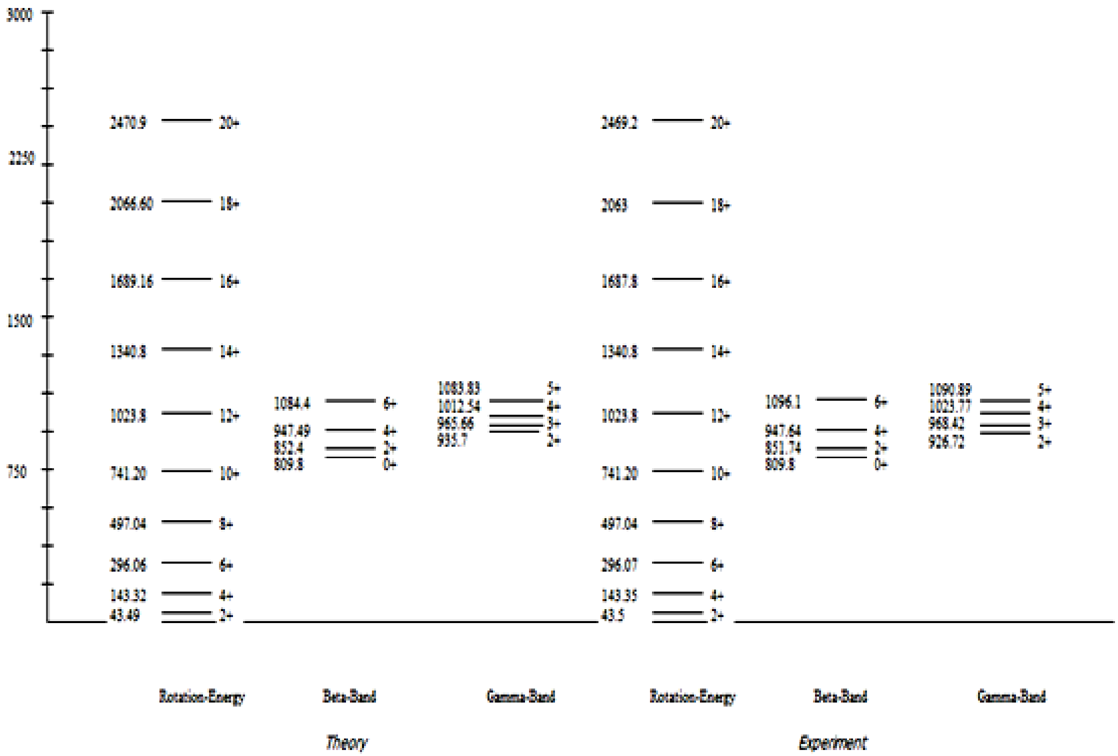

Fig.2-b (^{234}U)

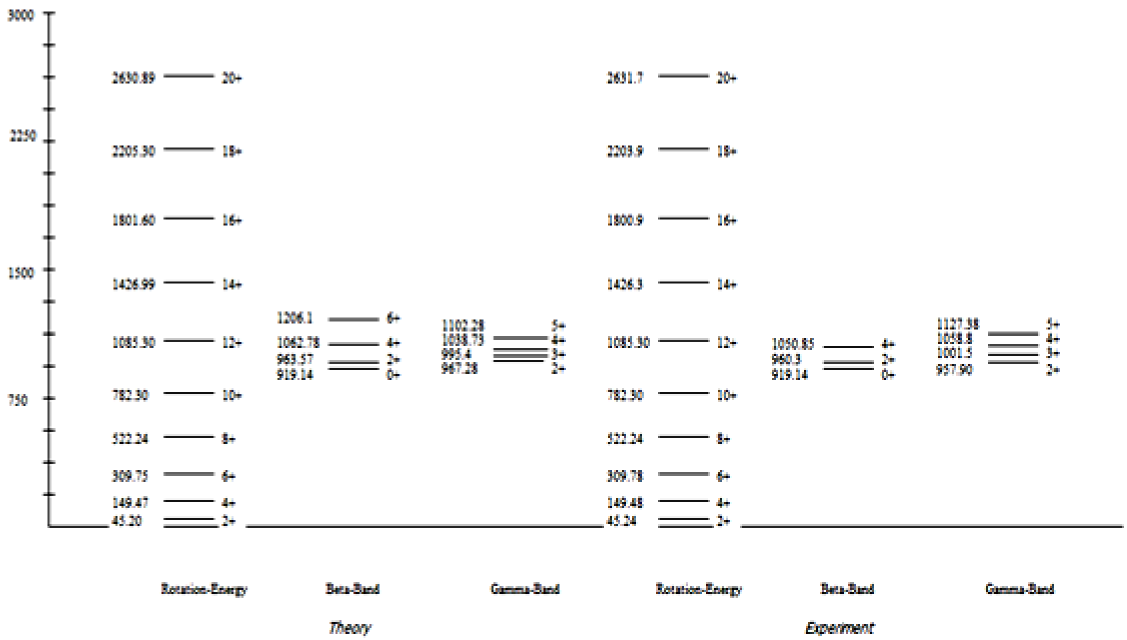

Fig.2-c (^{236}U)

Figs. 2-a, 2-b and 2-c. Variations of the rotational and vibrational energy eigenvalues of the uranium isotopes: ^{232}U, ^{234}U and ^{236}U with respect to the total angular momentum I.

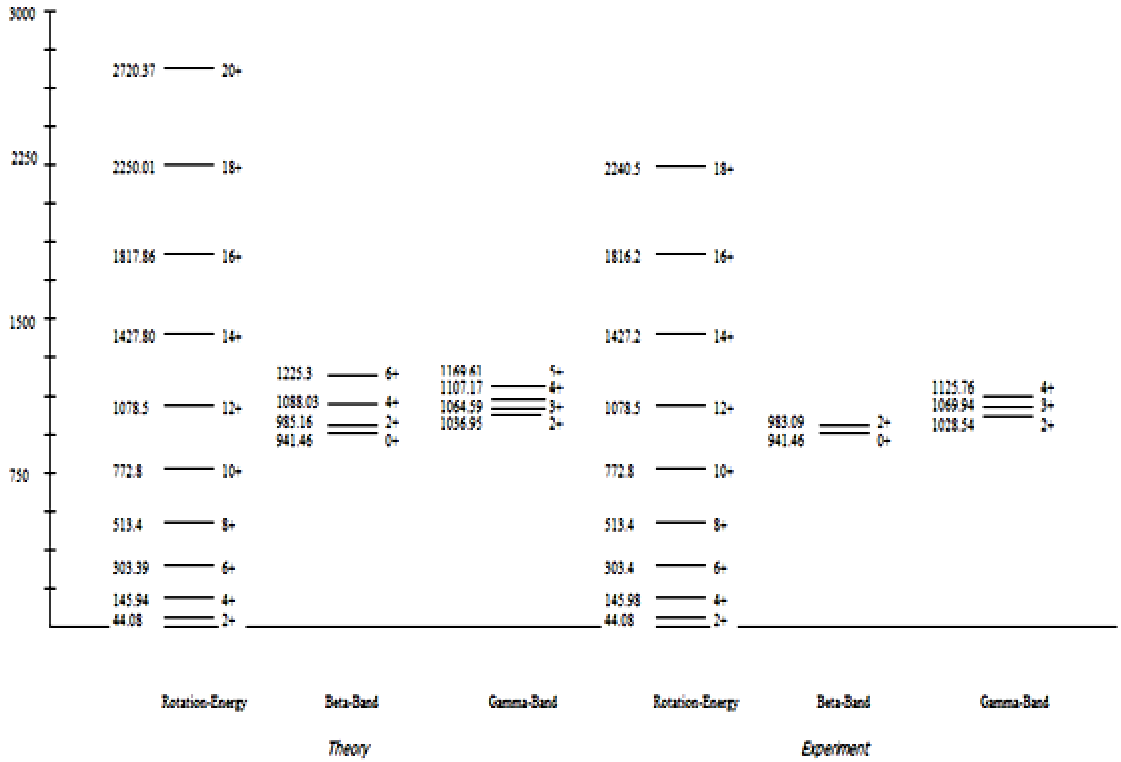

Fig. 3-a (^{238}Pu)

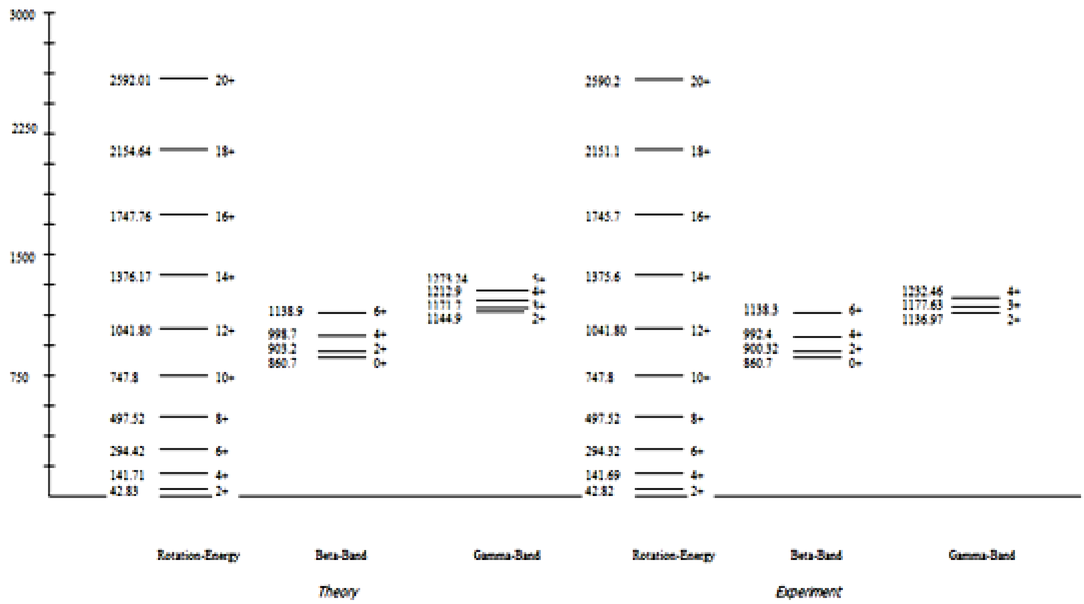

Fig. 3-b (^{240}Pu)

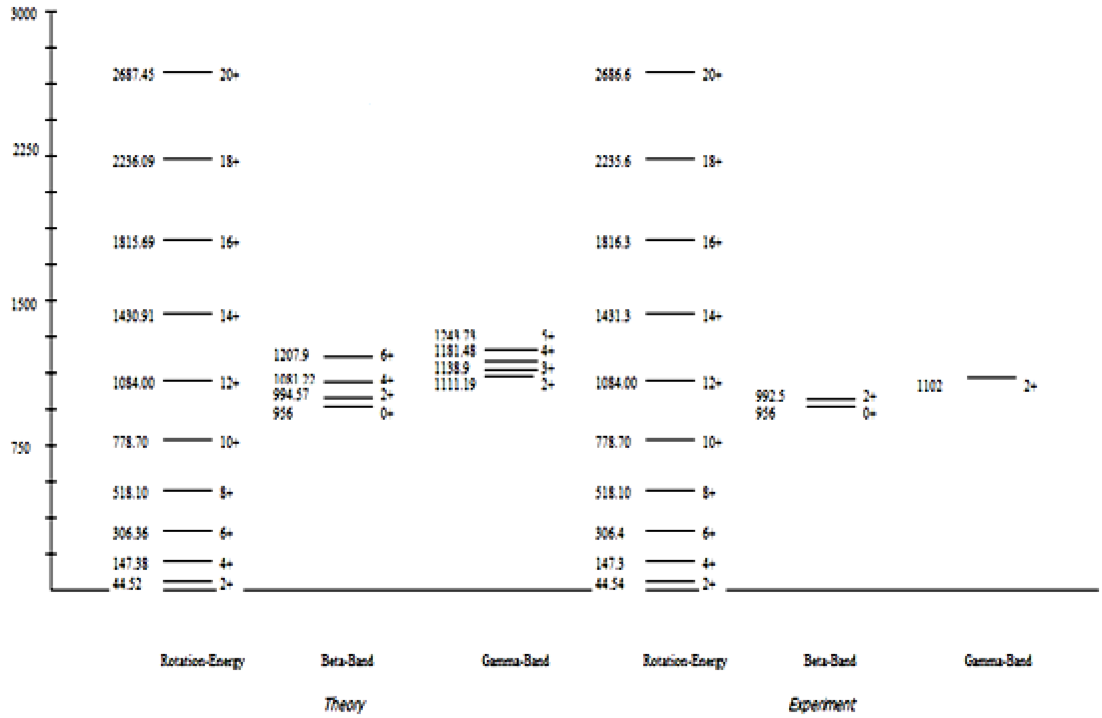

Fig. 3-c (^{242}Pu)

Figs. 3- a, 3-b and 3-c. Variations of the rotational and vibrational energy eigenvalues of the plutonium isotopes: ^{238}Pu, ^{240}Pu and ^{242}Pu with respect to the total angular momentum I.

From the obtained results we have seen that a new three-parameter formula for the rotational band of a well-deformed nucleus is suggested on the basis of the phenomenological Bohr Hamiltonian. In the derivation of this formula a small axial asymmetry and vibrational effects (including anharmonicity) have been taken into account. The agreement between the calculation using this formula and the observed ground bands (below band crossing) of all the actinide and the rare-earth deformed nuclei is astonishingly excellent, which perhaps implies that the formalism described in this paper may have some truth. However, we can indicate that the β- and γ-vibrations do not contribute much to the ground state band so we create new formulas for the β- and γ-vibrations band where the results of our calculation show good agreement with data in comparison with experimental data.

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