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Jeans's Criterion Using Various Cosmological Terms

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Abstract

Jeans's criterion concerning the gravitational instability of a quasi-infinite homogeneous medium is revisited. The original erroneous derivation is improved by considering the gravitational potential around an extremum. However, it is shown that the domain of valid approximation is about as large as the instability region, still leaving doubts about the correctness of the criterion. A derivation using a cosmological constant is obtained which avoids the inconsistency and recovers Jeans's criterion to a very good approximation. However, other problems arise. Moreover, when we use a sensible bilinear cosmological term then a kind of standard instability length results, not the Jeans's wavelength. Using a cosmological term quadratic in the density leads again to a good approximation of Jeans's criterion. Hence the precise form of the cosmological term turns out to be quite relevant.

The underlying physical mechanism of the instability is explained, confirming the right order of the Jeans's instability wavelength and Jeans's mass. The stability analysis of an infinite gravitational cylinder and of a plane-parallel medium support Jeans's result too. Some confirmations on the basis of celestial bodies (Earth, Sun, galaxy, cosmic tessellation) are briefly considered, including the effect of scaling. This leads to another interpretation of Jeans's wavelength: it is about (in fact slightly larger than) the diameter of celestial bodies in equilibrium and thus it expresses rather the stability of the body. The cosmic entropy concept is discussed in the light of Jeans's criterion.

Key words: Jeans's criterion, gravitational instability, star formation, cosmology, entropy

1 Introduction

In 1902 and in 1929 Jeans elaborated an argument due to Isaac Newton in his first letter to the Reverend Richard Bentley (10 December 1692). In a universe, homogeneously filled with matter, the gravitational forces balance each other in all directions. Hence, there is no net force acting on a piece of matter and a static equilibrium is manifest. However, a small perturbation may occur: is the homogeneous equilibrium stable or not? Newton expected instability and Jeans found it provided the wavelength of the perturbation exceeded a certain value depending on the parameters of the constellation.

However, Jeans made a mistake in his derivation, as he did not realize that a uniform potential is not allowed by the Poisson equation. Hence, there has been much unbelief and criticism about 'Jeans'swindle' in the literature. See e.g. Verheest (2000) who investigated whether the gravitation of the grains in an interstellar cloud might overcome the repulsive plasma forces. In a series of lectures (1986) the present author took the defense of Jeans, giving a physical explanation and showing that useful results could be obtained from the criterion. Here we recall some of these arguments and extend them further. Another confirmation of the criterion would be by eliminating the inconsistency by using a cosmological term. The idea here is that the precise form of the latter would not matter as it is very small, and it would lead to an approximation very close to Jeans's result. Indeed, this turns out to be the case when using the customary cosmological term. However, using an apparently more sensible cosmological term (bilinear in the density and the potential) leads to a different result as will be shown below. Surprisingly enough the use of a cosmological term quadratic in the density yields again Jeans's result to a good approximation. However, other problems arise. On the other hand, using the expansion of the universe to avoid the inconsistency leads effectively to the Jeans's wavelength, but gives a time evolution which is much too slow as shown by Lifshitz (1946) using general relativity, and by Bonnor (1965) using the Newtonian theory. We will confirm this objection in another way.

The plan of the paper is as follows. After this introduction an improved derivation of Jeans's instability is given in section 2. In section 3 we consider the domain of validity of the approximation. In section 4 we use first the usual cosmological constant attempting consistency, then we use a bilinear cosmological term and even one quadratic in the density. In section 5 the physical meaning of the criterion is exposed and several arguments in favor of it are given, e.g. dimensions and masses of some astronomical objects (e.g. Earth, Sun, galaxy and cosmic tessellation). The effect of contraction or expansion of a gravitating body is considered and the real meaning of Jeans's criterion is clarified. Finally, the concept of entropy on a cosmological scale is discussed in the light of Jeans's instability. The conclusions are in section 6.

2 Improved Derivation

2.1 Basic Equations

Like Jeans we consider an infinite homogeneous medium according to the Newtonian gravitation. The relevant equations are respectively the equation of continuity (expressing the conservation of mass), the equation of motion, the Poisson equation for the gravitational field and the equation of state:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \partial_t \rho + div \rho \bar{v} = 0, \tag{2.1}$$

$$\rho(\partial_t \bar{v} + \bar{v}.\nabla \bar{v}) = -\nabla p - \rho \nabla \varphi, \tag{2.2}$$

$$\Delta \varphi = 4\pi G \rho, \tag{2.3}$$

$$p = K \rho^{\Gamma}. \tag{2.4}$$

Here

$$G = (6.6726 \pm 0.0005) 10^{-11} m^3/kgs^2,$$

is the gravitational constant in SI units, K is a constant (= p_0/ρ_0^Γ) and Γ (Chandrasekhar, 1939) is taken as a constant whose value is usually between 1 (isothermal situation) and 5/3 (adiabatic case for 3 degrees of freedom), but may be considered as infinite for incompressible matter. φ is the gravitational potential, ρ is the mass density, p is the pressure and $\bar{v}$ is the velocity. $d/dt = \partial_t + \bar{v}.\nabla$ is the total derivative.

Notice that the equation for the gravitational field (the Poisson equation) is linear in φ and in ρ; however, the gravitational force is bilinear in φ and ρ. This may suggest that the cosmological term may be bilinear as well: see below.

2.2 Equilibrium

In view of the inconsistency in Jeans's derivation utmost care is required, following mainly Callebaut (1986). The index 0 is used to denote equilibrium quantities. For the equilibrium we put $\bar{v}_0 = 0$ in the basic equations:

$$\partial_t \rho_0 = 0, \rightarrow \rho_0(\bar{r}), \tag{2.5}$$

$$\nabla p_0 = -\rho_0 \nabla \varphi_0, \tag{2.6}$$

$$\Delta \varphi_0 = 4\pi G \rho_0, \tag{2.7}$$

$$p_0 = K \rho_0^\Gamma. \tag{2.8}$$

We restrict ourselves to the homogeneous static case

$$\rho_0 = Cst\ w.r.t.\ \bar{r}\ and\ t. \tag{2.9}$$

According to Eq. (2.8) we have that $p_0 = Cst$ as well. From Eq. (2.6) follows

$$\rho_0 \nabla \varphi_0 = 0. \tag{2.10}$$

If $\rho_0 = 0$ there is no matter and hence there can be no material changes. Hence we are forced to suppose $\nabla\varphi_0 = 0$, which implies that φ_0 is independent of $\bar{r}$. This suits our hypothesis of uniform ρ_0 and p_0 very well. Moreover, it satisfies our (and Newton's and Jeans's) intuitive feeling that any mass in an infinite homogeneous medium feels no net gravitational force because it is attracted equally well from all sides. However, $\nabla\varphi_0 = 0$ is incompatible with Eq. (2.7). This would lead us back to $\rho_0 = 0$, hence to an empty region, which would make our analysis senseless.

This problem of inconsistency is a serious one. It occurred also when Einstein applied his theory of general relativity (essentially a theory of gravitation) to obtain a cosmological model of the universe. It makes clear that the difficulty is not restricted to the Newtonian gravitation. Einstein added therefore a cosmological term to his theory to avoid an empty universe. Later he withdrew this term as a static equilibrium was no longer needed in view of the observed expansion of the universe. Nevertheless a cosmological term is still scientifically possible and several authors work on it. Its use has been suggested as a means to avoid the inconsistency in the Jeans's criterion (see below).

However, we shall first attempt to avoid the contradiction in Jeans's derivation by supposing that we consider φ_0 in the vicinity of an extremum (usually a minimum). There the first derivatives of φ_0 are very small and we may neglect them. The second derivatives, however, have not to be small there and thus $\Delta\varphi_0$ is not negligible and thus Eq. (2.7) may be satisfied. As an illustration one may think of a spherically symmetric volume (homogeneous or not). In its center there is no net force because all contributions balance each other perfectly in view of the spherical symmetry. Nevertheless Eq. (2.7) is satisfied in the center, because $\nabla\varphi \neq 0$ (but small) around it.

2.3 Perturbation, Linearisation, Elimination

As stated above we consider a region around an extremum of the potential. As usual in perturbation theory, we put any quantity, say X, equal to its equilibrium part (X_0) plus a perturbed part (X_1). We neglect the second order terms, i.e. products involving at least two perturbed parts. The linearized system of perturbed equations reads then (Callebaut1986)

$$\partial_t \rho_1 + div \rho_0 \bar{v}_1 = 0, \tag{2.11}$$

$$\rho_0 \partial_t \bar{v}_1 = -\nabla p_1 - \rho_0 \nabla\varphi_1 - \rho_1 \nabla\varphi_0, \tag{2.12}$$

$$\Delta\varphi_1 = 4\pi G \rho_1, \tag{2.13}$$

$$p_1 = \frac{\Gamma p_0}{\rho_0} \rho_1. \tag{2.14}$$

Here ρ_0 is supposed to be strictly constant so that $\nabla\rho_0 = 0$. Note that in Eq. (2.12) we did not neglect the term in φ_0 (as Jeans did), because it does yield a contribution in Eq. (2.15) through the second derivatives. We apply the divergence operator on the equation (2.12)

$$div(\rho_0 \partial_t \bar{v}_1) = -\Delta p_1 - \rho_0 \Delta\varphi_1 - \rho_1 \Delta\varphi_0 - \nabla\rho_1.\nabla\varphi_0. \tag{2.15}$$

Here, we will neglect the term $\nabla\rho_1.\nabla\varphi_0$ in Eq. (2.15) in agreement with our approximation that in the vicinity of an extremum of φ_0 its first derivatives are small, so that on multiplication with a first order term it gives rather a second order term. Using Eqs. (2.11), (2.13 - 15) and (2.7) allows to eliminate $\bar{v}_1, p_1, \varphi_1$ and φ_0 to obtain a partial differential equation of second order in ρ_1

$$\partial^2_{tt} \rho_1 = v_s^2 \Delta\rho_1 + 8\pi G \rho_0 \rho_1. \tag{2.16}$$

Here we have put $v_s^2 = \Gamma p_0/\rho_0$ with v_s the sound velocity. Note that Jeans obtained 4π instead of 8π as coefficient in the last term because he omitted the term $\rho_1\Delta\varphi_0$ in Eq. (2.15). The resemblance with the equation for pure sound waves is obvious as can be seen by putting $G = 0$; however, Eq. (2.16) has an additional term and thus it is no more homogeneous in the derivatives. It is still linear and homogeneous in the unknown function, ρ_1, as we want it to be in view of the linearisation of the system.

2.4 Fourier Analysis and Dispersion Equation

We do not have a complete solution at hand like in the case of the pure sound equation. A Fourier analysis is indicated, if not unavoidable. We put

$$\rho_1 = C_1 \rho_0 e^{i(\omega t + \bar{k}.\bar{r})}, \tag{2.17}$$

with C_1 a dimensionless constant which is arbitrary, but supposed small in view of the linearisation of the system. Here $k = 2\pi/\lambda$ is the wavenumber of the perturbation with wavelength λ (in Cartesian coordinates we have $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$) and $\omega = 2\pi\nu$ is the angular frequency, where ν is the frequency; if ω is purely imaginary, $\omega = i\sigma$, it determines the growth rate σ. Substituting Eq. (2.17) in Eq. (2.16) yields

$$\omega^2 = v_s^2 k^2 - 8\pi G \rho_0. \tag{2.18}$$

This is the famous dispersion relation of Jeans (1902), however, corrected, i.e. with a factor 8π instead of his 4π, which, however, is a change of minor importance. The dispersion equation should represent the gravitational waves and instabilities in a Newtonian gravitating medium which is infinite and homogeneous in all directions; we suppose that it is valid for finite distributions as well in view of our approximation around an extremum of the potential.

2.5 Critical Wavelength, Critical Mass and Maximum of Growth Rate

The angular frequency ω becomes zero for the so-called wavenumber of Jeans

$$k_J = \frac{\sqrt{8\pi G \rho_0}}{v_s} = 2\rho_0 \sqrt{\frac{2\pi G}{\Gamma p_0}}. \tag{2.19}$$

For this critical k_J there is an exchange of instabilities, i.e. for $k > k_J$ we have $\omega^2 > 0$, which corresponds to oscillations, while for $k < k_J$ we have that $\omega^2 < 0$, which means instabilities. With k_J corresponds the (critical) wavelength of Jeans

$$\lambda_J = \frac{1}{\rho_0} \sqrt{\frac{\pi \Gamma p_0}{2G}}. \tag{2.20}$$

Note as yet that k_J and λ_J depend on the equilibrium parameters p_0 and ρ_0, as well as on the gravitational constant G and on Γ, which is a constant related to the state of the matter.

For $k < k_J$ we have instability instead of a wave. The maximal growth rate σ_m is for $k = 0$, what would mean an infinite wavelength,

$$\sigma_m = \sqrt{8\pi G \rho_0}. \tag{2.21}$$

We note that σ_m depends on ρ_0 and the gravitational constant only, as opposed to λ_J, which depends on p_0 and Γ too. It may be remarked here that in many gravitational problems σ_m^{-1} plays the role of a characteristic time.

The mass enclosed by a cube with side λ_J is

$$M_J = \rho_0 \lambda_J^3 = \frac{1}{\rho_0^2}\Big(\frac{\pi\Gamma p_0}{2G}\Big)^{3/2} = \frac{1}{\sqrt{\rho_0}}\Big(\frac{\pi\Gamma k_B T_0}{2mG}\Big)^{3/2}. \tag{2.22}$$

Here T_0, the temperature of the equilibrium, is introduced; it is supposed uniform like the other equilibrium quantities; m stands for the mass of the molecules in the medium and n is the number density ($nm = \rho$); $k_B = 1.38... \times 10^{-23}$ J/K is the Boltzmann constant; $p_0 = n_0 k_B T_0$. M_J is called the Jeans's mass (up to a factor of order unity, according to the various authors). This is about the smallest mass which can be generated by instability. (Using spherical coordinates allows a mass which is roughly half M_J, see below.) However, as the growth rate for λ_J is zero, the chances for this mass to be formed are very small, as other instabilities grow faster and thus may prevail. The more probable masses to be found will generally be somewhat larger than M_J. For $k \to 0$ one would expect very large masses; however, these have a large probability to undergo anew the Jeans's instability, which leads then in turn again to masses like Jeans's mass or somewhat bigger.

However, more important than the precise coefficient which may be placed in front of M_J, is the fact that $\sqrt{\rho_0}$ appears in the denominator of Eq. (2.22): M_J decreases with increasing ρ_0 for a given temperature. This is at first sight surprising, but it is understandable: smaller ρ_0 requires a larger volume to reach the same (critical) mass, but since the distances have increased a still larger mass, and thus a still larger volume, is needed to compensate for the pressure opposing the contraction caused by the instability. This is so although p_0 has decreased due to the decrease in ρ_0. If p_0 is kept unchanged (e.g. by compensating with a larger temperature) then the effect of a change in ρ_0 is even much more pronounced: $M_J \propto \rho_0^{-2}$.

Note that M_J is proportional to $T_0^{3/2}$. This is understandable: a larger temperature yields a larger pressure and hence a larger mass is needed to be able to overcome the opposition of this pressure against the gravitational contraction. Now $\lambda_J \propto \sqrt{p_0}$ and since $p_0 \propto T_0$ we have the power 3/2 for T_0 in Jeans's mass. It follows that the critical mass of Jeans is fairly dependent on the equilibrium temperature. It depends on the molecular mass too, but in the formation of stars this is dominantly the mass of the hydrogen atom ($m = 1.67... \times 10^{-27}$ kg).

2.6 Spherical Coordinates

Let us reconsider Eq. (2.16), however, now in spherical coordinates and assuming spherical symmetry for the perturbation. Hence, instead of Eq. (2.17) we consider now the solution

$$\rho_1 = \frac{C_1}{r}\rho_0 e^{i(\omega t + k_r r)}, \tag{2.23}$$

or rather

$$\rho_1 = \frac{C_1}{r}\rho_0 e^{i\omega t} sin k_r r, \tag{2.24}$$

to avoid the singularity in the origin. We have similar conventions as in Eq. (2.17), e.g. k_r is now the wavenumber in spherical coordinates. However, the dispersion relation reads exactly the same as Eq. (2.18) with k_r instead of k. Actually, the largest unstable wavenumber is again k_J and the corresponding smallest wavelength is again λ_J. The central mass corresponds to the central sphere with radius $\lambda_J/2$

$$M'_J = \frac{4\pi\rho_0}{3}(\lambda_J/2)^3 \cong \frac{M_J}{2}. \tag{2.25}$$

Hence, the critical wavelength is the same, but the smallest mass is half as large as given first when using Cartesian coordinates. The nearest shell surrounding the central sphere (between radii $\lambda_J/2$ and $3\lambda_J/2$) has 7 times more mass. However, it is liable to instabilities which may subdivide it into e.g. 4 masses, which are again about M_J.

Let us reconsider the neglected term $\nabla\rho_1.\nabla\varphi_0$ in Eq. (2.15), taking spherical symmetry for the perturbation around the extremum of φ_0. Then $\nabla\rho_1$ and $\nabla\varphi_0$ are parallel, so the term may be small, but not zero. This term will generate a term in ik_r in the dispersion relation, thus ω will be complex (for real k_r), representing a mixture of an oscillation and an instability. It may then be necessary to consider the nonlinear theory to reveal whether there is ultimately instability and what dimensions correspond to it. On the other hand it is maybe possible to compensate for this term by using an appropriate (bilinear) cosmological term.

3 Domain of Validity of our Approximation

We may wonder how good our approximation is, i.e. neglecting the first derivatives of φ_0 while keeping Eq. (2.7). The inconsistency may be avoided in a point or a curve and even on a surface where the potential has an extremum. It can be avoided approximately in a spatial region surrounding an extremum of the potential, as we suggested when starting the analysis. However, how large is that spatial region? More precisely: how large is that region as compared to λ_J (in the direction of $\nabla\varphi_0$ and perpendicular to it)?

Let us estimate a typical scale length for a gravitational body in equilibrium under the influence of pressure and gravity. Call this length L; that means that over a length L the values of $\varphi_0(\bar{r})$ and $p_0(\bar{r})$ have changed about from a maximum to a minimum, at least in the direction of $\nabla\varphi_0$. Let us consider the equilibrium equations (2.6) and (2.7). We replace the differentials as follows

$$dp_0 = p_{max} - p_{min},\ \ d\varphi_0 = -(\varphi_{max} - \varphi_{min}),\ dr = L. \tag{3.1}$$

Then, at least in the direction perpendicular to the equi-potential surfaces, one has

$$p_{max} - p_{min} \approx \rho_0(\varphi_{max} - \varphi_{min}), \tag{3.2}$$

$$\varphi_{max} - \varphi_{min} \approx 4\pi G\rho_0 L^2. \tag{3.3}$$

The replacement of dr by L is without much error in Eq. (3.2) as it happens on both sides; however, in Eq. (3.3) where the second differential is replaced by L^2 the approximation may be rougher. Elimination of $\varphi_{max} - \varphi_{min}$ yields

$$p_{max} - p_{min} \approx 4\pi G\rho_0^2 L^2. \tag{3.4}$$

Now we replace p_{max} by p_0. As $p_{max} > p_{max} - p_{min}$, we obtain for the typical scale length

$$L \leq \sqrt{\frac{p_0}{4\pi G\rho_0^2}}. \tag{3.5}$$

Comparison with Jeans's wavelength yields

$$\lambda_J \approx \pi\sqrt{2\Gamma}L, \tag{3.6}$$

even when taking the largest value for L allowed by inequality (3.5). The coefficient of L in Eq. (3.6) is about 5, even when we take the lowest value for Γ, meaning that λ_J is some 5 times larger than L. Now L is certainly larger than the length over which we may consider the medium as homogeneous (in a direction perpendicular to an equi-potential surface. Note that there is no objection along equi-potential lines or surfaces, which may be useful e.g. for spherical shells as mentioned in section 2.6.) Thus, according to this estimate, λ_J may be several times larger than the length over which the medium may be considered as homogeneous! We admit that our approximation here is a rough one, but refined calculations (Callebaut, 1986) confirmed this fully. Hence, the analysis leading to λ_J and to M_J is still subject to serious criticism, even when applied around an extremum of the potential. A real safe derivation should consider a very extended region of mass and taking into account $\nabla\rho$ and $\nabla\varphi$. However, this yields involved differential equations. Nevertheless, it seems possible, to handle the equations with minor approximations, for spherical shells of a finite configuration with spherical symmetry. For spherical shells the density, pressure and potential are nearly constant, varying in the radial direction only.

It may be noted that we may interpret L as the gravitational Debye length λ_{DG}. In plasmas (and similarly in electrolytes) the Debye length $\lambda_D = \sqrt{\frac{\epsilon k_B T}{ne^2}} = \frac{\sqrt{\epsilon p_0/2}}{\rho_e}$ is the distance over which the plasma may be considered as quasi-neutral (e is the charge of the electron, $\rho_e = ne$ is the charge density of the electrons, ϵ is the permittivity ($\epsilon_0 = 8.86 \times 10^{-12}$ F/m in free space). We have put $nk_BT = p_0/2$ in view of the pressure exerted by the positive charges). If we replace ϵ by $1/4\pi G$ and ρ_e by ρ_0 we obtain formally

$$\lambda_{DG} = \frac{1}{\rho_0}\sqrt{\frac{p_0}{8\pi G}} \cong \sqrt{2}L. \tag{3.7}$$

λ_{DG} has no meaning as a shielding length in a sense like in electro-magnetics, but may be interpreted as L, roughly the distance over which the medium is uniform. Thus λ_J corresponds to λ_{DG} as well, up to a factor about 5.

4 Derivation using Various Cosmological Terms

When Einstein tried in 1917 to apply his monumental theory of gravitation (usually called the general theory of relativity, 1915) to cosmology he hit on the same difficulty concerning the equilibrium (probably without knowing the Jeans's criterion). Even in a curved space the only possible homogeneous and static universe allowed by his original equations of 1915 had to be an empty one (i.e. $\rho_0 = 0$).

To avoid the empty universe Einstein introduced his so-called cosmological term in his theory of gravitation. The new gravitational field equation reads then in tensor form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{4.1}$$

Here $R_{\mu\nu}$ is the Ricci tensor, $R = R^{\mu}_{\mu}$ is its contraction, $g_{\mu\nu}$ is the metrical tensor, $T_{\mu\nu}$ is the energy-momentum tensor, $\kappa = 8\pi G/c^4$ (with c the velocity of light in vacuo) and Λ is the cosmological constant, which was zero in his 1915 theory. The addition of this cosmological term ($\Lambda g_{\mu\nu}$) was the simplest possible extension of Einstein's 1915 theory without drastically changing the whole setup of his theory.

The introduction of a cosmological term in the gravitational field equation may avoid the inconsistency in the analysis leading to Jeans's criterion. Moreover, as a cosmological term is extremely small it may be hoped that the Jeans's result may be recovered up to negligible corrections. That is indeed the case when using the customary cosmological term (the simplest one). However, other (in our opinion more adequate) cosmological terms cast some doubts on this hope.

4.1 Customary Cosmological Term

In order to simplify the matter we shall explain here the essential features by using an approximation like the one leading to the Newtonian theory. Thus we obtain from Eq. (4.1) the analogue of the Poisson equation

$$\Delta\varphi - \Lambda\varphi = 4\pi G\rho. \tag{4.2}$$

Now we can have a homogeneous universe in equilibrium with $\nabla\varphi_0 = 0$ and thus $\Delta\varphi_0 = 0$. Eq. (4.2) yields then

$$\Lambda\varphi_0 = -4\pi G\rho_0. \tag{4.3}$$

This fixes the (normally arbitrary) constant of the potential: proportional to the (average) density in the universe, a reasonable result. In the Newtonian theory this constant was arbitrary, what raises questions as to its meaning. The cosmological term prevents at the same time the universe from being empty, i.e. ρ_0, here ρ_U, is now different from zero. It may be noted that due to the present cosmological term the Newtonian potential at a distance r from a point mass M is changed to

$$\varphi(r) = \frac{GM}{r}e^{-r\sqrt{\Lambda}} + \varphi(0), \tag{4.4}$$

A potential like this one, involving an exponential, was already proposed by Neumann in 1895, however, rather ad hoc to avoid ever increasing potentials in cosmology. In fact the amount of matter in a homogeneous medium increases like r^3, so the corresponding potential would increase like r^2 were it not that the exponential forces it to approach zero at distances that are many times $1/\sqrt{\Lambda}$. Note by the way that later the Debye potential for electrolytes and plasmas had the same structure and was strictly derived; and still later the hypothetical Yukawa potential for nuclear forces had again a similar structure. In fact

Λ is supposed to be extremely small so that its influence in e.g. the solar system is not yet measurable: the exponential in Eq. (4.4) is practically unity even for several lightyears. However, for cosmological distances the potential approaches zero. This means that the usual Newtonian *long distance* potential is erased by the exponential at cosmical distances. This allows an equilibrium configuration for a very large (quasi-infinite) gravitating homogeneous medium.
We keep most of the basic equations of section (2), except those which are replaced by Eqs. (4.2) and (4.3). Eq. (2.12) is replaced by

$$\rho_0 \partial_t \bar{v}_1 = -\nabla p_1 - \rho_0 \nabla \varphi_1, \tag{4.5}$$

as φ_0 is now constant. Thus Eq. (2.15) is replaced by

$$div(\rho_0 \partial_t \bar{v}_1) = -\Delta p_1 - \rho_0 \Delta \varphi_1. \tag{4.6}$$

The perturbed field equation reads

$$\Delta \varphi_1 - \Lambda \varphi_1 = 4\pi G \rho_1. \tag{4.7}$$

Eq. (4.6) yields, upon elimination of $\bar{v}_1, p_1$ and $\Delta\varphi_1$

$$\partial^2_{tt} \rho_1 = v_s^2 \Delta \rho_1 + (4\pi G \rho_1 + \Lambda \varphi_1)\rho_0. \tag{4.8}$$

In order to have an equation in ρ_1 only, we have to eliminate φ_1. To do so we extend the Fourier analysis, Eq. (2.17), to φ_1. Eq. (4.7) yields

$$-(k^2 + \Lambda)\varphi_1 = 4\pi G \rho_1. \tag{4.9}$$

Using Eq. (4.9) to eliminate φ_1 we obtain from Eq. (4.8) the improved dispersion equation

$$\omega^2 = v_s^2 k^2 - 4\pi G \rho_0 \frac{k^2}{k^2 + \Lambda}. \tag{4.10}$$

For $\Lambda \to 0$ Eq. (4.10) yields the original result of Jeans; its limit differs from our result by the rather unimportant factor 2. The dimensions of a star are of the order of 10^9 m, the corresponding k-values are then respectively about 10^{-8}/m. However, Λ is estimated to be still several orders of magnitude smaller. The dimensions of a galaxy are of the order of $10^{20\ to\ 22}$m and the k-values around $10^{-19\ to -21}$/m.
Thus the Jeans's wavelength and corresponding critical mass are correctly derived to a very great approximation using the field equation with the cosmological constant. However, there is a drawback. For $k^2 < \Lambda$, i.e. for extremely large wavelengths, we may approximate Eq. (4.10) by

$$\omega^2 = \left(v_s^2 - \frac{4\pi G \rho_0}{\Lambda}\right)k^2. \tag{4.11}$$

Neglecting v_s^2 yields $\sigma \cong k\sqrt{\frac{4\pi G\rho_0}{\Lambda}}$. With $k << \sqrt{\Lambda}$ the growth rates become very small. This is totally different from the situation without using a cosmological constant, where the maximum growth rate occurred for $k = 0$. Due to the cosmological constant those gigantic instabilities are indeed larger than Jeans's wavelength but they grow extremely slowly, which may be a problem for the formation of galaxies or clusters of galaxies and for the tessellation of the universe. The estimated age of the universe is 13.7 billion years. It may be recalled that Lifshitz (1946) and Bonnor (1965) when using the expansion of the universe hit on a similar problem: they recovered the Jeans's instability wavelength, but the growth of the instabilities was too small, (actually a power law and not an exponential) so the global result is not yet satisfactory.

Three reflections/objections arise. *First*, why do we need a term, which is expected to play a role on a cosmological scale only, to obtain aspects like the dimensions of planets or stars or even galaxies? Physically we have the (risky!) intuitive feeling that Jeans's derivation is a satisfactory approximation, so that any reasonable mathematical way to avoid the inconsistency is greatly welcomed. However, it still looks somewhat suspicious. *Second reflection*: we have avoided an inconsistency, but introduced another one. In fact we argue as if the whole universe is filled with the equilibrium density ρ_0, while the universe has a much, much lower average density than the clouds which generate planets, stars or galaxies. Arguing that we consider a very extended region with density ρ_0 surrounded by a quasi-infinite region with the density of the universe is not convincing, but will be applied with some subtlety later.
Third reflection: the cosmological term under consideration is far from firmly established. Einstein himself withdrew the cosmological term as soon as it was no more absolutely needed. What if another cosmological term is used? Again, intuitively (but risky!), we expect that its ultimate influence will be very small as illustrated in the present section for the customary cosmological constant. However, let us see another cosmological term.

4.2 Bilinear Cosmological Term

An objection against the customary cosmological term is that it is fairly ad hoc: the main argument is that it is the simplest possibility. It has not a physical basis. E.g. the factor r^{-2} in the Newtonian force has a clear geometrical meaning: the surface of a sphere through which the field passes is proportional to r^2, hence it is understandable that the intensity of the force must decrease as r^{-2} in order to have a total conservation of flux of the force, which seems a plausible demand. On the other hand, there is no geometrical or physical argument for the exponential factor in Eq. (4.4). No wonder that Einstein withdrew his cosmological term as soon as it was no more a necessity, although many scientists still believe firmly in a cosmological term of one kind or another. Actually the equation of motion (2.2) is bilinear in φ and ρ: maybe the gravitational field equation is bilinear in φ and ρ too? That may physically be interpreted that the potential experiences an attenuation when passing through matter. As the potential does exert a force on matter through its gradient an attenuation due to the matter seems a much fairer possibility than the customary cosmological term (Callebaut and Khater, 2005 and Callebaut, Karugila and Khater, 2005). Hence we replace Eq. (4.2) by

$$\Delta\varphi - \Lambda'\rho\varphi = 4\pi G\rho. \tag{4.12}$$

Here Λ' is another cosmological constant. Of course, the masses in the universe are distributed in discrete lumps as ions, or stars or even galaxies, but in view of the large extensions we may use the approximation of an average density. For the equilibrium of an infinite homogeneous medium we have then

$$-\Lambda'\varphi_0 = 4\pi G. \tag{4.13}$$

This result is surprising as it means that φ_0 is just the ratio of two universal constants. Hence it is a universal constant itself and not (directly) related to e.g. the mass density or to the radius of the universe or so. (Remember that the harmonic oscillator in quantum mechanics has not zero as lowest energy level - although this is a totally different situation. Moreover, the lowest level of the harmonic oscillator is still related to the oscillator.) In empty space we recover the Laplace equation and for the field around a point mass situated in the homogeneous density medium or by vacuum we recover the usual Newtonian expression. This is not a satisfactory feature as we were hoping to obtain a potential which varies like $r^{-1}e^{-r\sqrt{\Lambda'\rho_0}}$ in analogy with the customary cosmological case, see Eq. (4.4), now having the desired weakening as due to the density. .

Perturbing and linearizing Eq. (4.12) yields

$$\Delta\varphi_1 - \Lambda'\rho_0\varphi_1 = 0, \tag{4.14}$$

where we have made use of Eq. (4.13) what unfortunately eliminates ρ_1 as well and thus prevents us to obtain a relation between ρ_1 and φ_1. Now Eq. (4.8) yields

$$\partial^2_{tt}\rho_1 = v_s^2\Delta\rho_1 + \Lambda'\rho_0^2\varphi_1. \tag{4.15}$$

Using a Fourier analysis and applying it to Eqs. (4.14) and (4.15) yields respectively

$$k^2 = -\Lambda'\rho_0, \tag{4.16}$$

$$(\omega^2 - v_s^2k^2)\rho_1 = k^2\rho_0\varphi_1, \tag{4.17}$$

Eq. (4.16) fixes k independent of ω, a surprising feature. Moreover, if $\Lambda' > 0$, then k must be imaginary, corresponding to a different type of wave or instability. Eq. (4.17) links ρ_1 and φ_1, but does not give a proper dispersion relation. There is only one k while ω is arbitrary, however, once ω is chosen the ratio φ_1/ρ_1 is fixed and vice versa; in fact this ratio replaces the usual arbitrariness of k. However, for appropriate ratios one may obtain imaginary $\omega's$, hence instability.

$$\omega^2 = k^2\big(v_s^2 + \rho_0\varphi_1/\rho_1\big) = -\Lambda'\rho_0\big(v_s^2 + \rho_0\varphi_1/\rho_1\big). \tag{4.18}$$

Corresponding to the wavenumber in Eq. (4.16) we obtain the unique wavelength of the perturbation

$$\lambda = \frac{2\pi}{\sqrt{-\Lambda'\rho_0}}, \tag{4.19}$$

provided Λ' is negative. Just one instability wavelength is a surprising feature in perturbation theory. This does not look suitable for stars and planets, as there is no influence of the pressure or the temperature. Maybe some structures correspond to it, but we need first an approximation of Λ'. If we take for $\rho_0 = 10^{-27}$ kg/m^3 and for $\Lambda' \cong -10^{-25}$m/kg (compare with the customary estimation of Λ), then we find $\lambda \approx 10^{26}$m, which is not far from the dimensions of the universe and may be related to the tessellation of the universe. However, we do not see much of a link with Jeans's criterion. This shows clearly that the precise cosmological term does matter and does not necessarily lead to an asymptotic approximation of Jeans's result.

4.3 Alternative Bilinear Cosmological Term

To avoid the second objection formulated at the end of section (4.1) we may consider the following field equation

$$\Delta\varphi - \Lambda'(\rho_0 + \rho_U)\varphi = 4\pi G(\rho_0 + \rho_U). \tag{4.20}$$

Here ρ_0 is the density of the medium which is supposed homogeneous over a large domain (i.e. large as compared to any wavelength which may occur) but by far not the infinite domain. On the other hand ρ_U corresponds to the (average) density in the cosmos. This density may be partially due to radiation. In this way we avoid more or less an inconsistency, mentioned as the second reflection at the end of section (4.1), but the results are not really different from the previous sections. Intuitively this approach gives a good feeling because the gradient of the density can be very small by using a very extended region. However, the larger the volume, the larger the pressure in it in order to have an equilibrium. The pressure occurs in the Jeans's expression. Thus λ_J increases with the extension. This becomes much clearer in section 5.4.

It is obvious that the second objection of section (4.1) does not apply to the universe itself. Hence, Jeans's criterion may be applicable to the universe and may explain the tessellation or the expansion of the universe.

4.4 Other Alternatives

1) **Attempt using $\rho^2\varphi$ as cosmological term.**

In the theory with the customary cosmological constant the field around a point mass contains an exponential having $\sqrt{\Lambda}$ in the exponent, see Eq. (4.4). Hence, we consider briefly the following field equation

$$\Delta\varphi - \Lambda''\rho^2\varphi = 4\pi G\rho, \tag{4.21}$$

thus expecting around a point mass an exponential containing linearly the density (which is constant as well when averaging in the universe). Moreover, in this way we avoid the previous situation in which one wavelength only is allowed. We do not have yet other physical reasons to suppose the square of the density in the field equation; in fact the square is at least surprising.

In a uniform medium we obtain

$$\varphi_0 = -\frac{4\pi G}{\rho_0\Lambda''}. \tag{4.22}$$

Perturbation of Eq. (4.21) gives

$$\Delta\varphi_1 - \Lambda''\rho_0^2\varphi_1 = -4\pi G\rho_1. \tag{4.23}$$

Notice the minus sign on the right hand side! Using a Fourier analysis yields

$$(k^2 + \Lambda'' \rho_0^2)\varphi_1 = 4\pi G \rho_1. \tag{4.24}$$

Eq. (2.16) yields now

$$\omega^2 = v_s^2 k^2 - \frac{4\pi G \rho_0 k^2}{k^2 + \Lambda'' \rho_0^2}. \tag{4.25}$$

This reproduces again Jeans's dispersion relation when $\Lambda'' \to 0$. Again, the growth rates become very small when $k^2 << \Lambda'' \rho_0^2$, which rises a problem in view of the finite time of existence of the universe.

2) **Expanding universe**

This avoids the equilibrium problem, but asks for a more involved perturbation of the steady state (the expansion is opposed to the gravitational contraction of an instability). Moreover, one faces the choice of neglecting (or not) the non-linear terms in the velocity. Lifshitz investigated in 1946 the consequences of the expansion of the universe on the gravitational stability, using the general theory of relativity of Einstein. Later, in 1957, Bonnor did the same for the Newtonian approximation. In both cases they obtained instability for wavelengths larger than λ_J, however, the growth is not exponential, but with a power of t as mentioned above. That is too slow to allow galaxies or even stars to be born and to develop as the estimated lifetime of the universe is about 13.7×10^9 years. The growth is too slow even when taking into account that the intergalactic distances increase due to the expansion. The nonlinear approach does not promise much help.

Working with the cosmological constant is a totally different approach. Nevertheless, we obtained similar objections: the instabilities are too slow, although still yielding an exponential evolution.

5 Physical Arguments in Favor of Jeans's Criterion

In spite of all the objections made, mainly the problem of a hypothetical equilibrium of a quasi-infinite gravitating medium, we do believe in the usefulness of Jeans's criterion, although we will adapt the interpretation. Here we give three physical arguments which prove the utility of λ_J and M_J.

5.1 Gravitation is a Long Range Force which Ultimately Wins over the Pressure

Indeed, consider a wavelength λ in spherical coordinates and spherical symmetry and the correspondingly mass around the center. If we increase this wavelength, then the gravitational force will ultimately become sufficiently large to overcome the pressure and to generate an instability by contraction. In fact, the pressure is the agent which opposes the compression, which is generated by the gravitation. However, the pressure in a homogeneous medium is constant over long distances (at least as compared to λ_J), to which the (beginning) wave adds little. On the other hand the gravitational force does increase with the wavelength since it is proportional to the enclosed perturbed mass λ^3, divided by the distance squared, i.e. $\cong \lambda^2$. Hence the gravitational force increases like λ at the border of the enclosed mass and for a sufficiently large λ it will overcome the pressure which is more or less independent of the wavelength. To make the argument a bit more quantitative we consider a spherical blob of matter of radius $\lambda/4$ around the center under consideration: its mass is (omitting the perturbation) $4\pi\rho_0(\lambda/4)^3/3$; the mass in the spherical shell around it between radii $\lambda/4$ and $\lambda/2$ is $4\pi\rho_0\lambda^3(\frac{1}{8}-\frac{1}{64})/3$. The gravitational attraction between those two masses is $7G(\pi\rho_0)^2\lambda^4/144$, where we have approximated their average distance as $\lambda/4$ and where we neglect the surroundings (everything farther than $\lambda/2$) because of the spherical symmetry. We expect that this mass will contract when it overcomes the pressure, approximated by p_0, multiplied by the surface of a sphere with radius $\lambda/4$:

$$7G(\pi\rho_0)^2\lambda^4/144 \geq p_0\pi\lambda^2/4. \tag{5.1}$$

This leads to

$$\lambda^2 \geq \frac{36p_0}{7\pi G\rho_0^2} \cong \frac{5p_0}{3G\rho_0^2} \tag{5.2}$$

what corresponds to λ_J up to a factor of the order unity, see Eq. (2.20). All this means that when the gravitating masses enclosed in the compressed parts of a perturbation are sufficiently large (be it 10 M_J or more) then the instability will manifest itself.

In fact this qualitative argument (that gravity ultimately wins against the pressure provided the scale is increased sufficiently) can be used to show that (small) inhomogeneities and thus the corresponding gradients of pressure, density or gravitational potential will not affect much the Jeans's length, at least if the inhomogeneity is neither too strong nor extending (more or less monotonically) over a too large region.

5.2 Correct Infinite Media

Another strong argument in favor of the λ_J is that the stability analysis of correct equilibria like the infinite homogeneous gravitational cylinder or the infinite plane-parallel gravitating medium confirms λ_J up to a factor of the order unity. For an infinite gravitational cylinder of incompressible, homogeneous material Chandrasekhar and Fermi (1953), thinking of a model for an arm of a spiral galaxy, obtained for the critical wavelength

$$\lambda_{CF} = 2\pi R/1.0668... \tag{5.3}$$

where R is the radius of the cylinder. However, the geometrical factor R may be expressed in terms of the density ρ_0 and the pressure at the axis of the cylinder $p_a = \pi G\rho_0^2 R^2$, yielding

$$\lambda_{CF} = \frac{2}{1.0668...}\sqrt{\frac{\pi p_a}{G\rho_0^2}}, \tag{5.4}$$

which corresponds with λ_J up to a factor of about 2. The pressure at the axis is entirely due to the gravitation of the cylinder in equilibrium (no external pressure).

For the plane-parallel homogeneous, incompressible gravitating medium of thickness $2d$ (Callebaut, 1972), thinking of flat disc (e.g. a model of a spiral galaxy), obtained the following critical wavelength, separating stable oscillations from instabilities

$$\lambda_c = \frac{2\pi d}{0.6392...}. \tag{5.5}$$

Using the pressure in the middle of the plane-parallel medium (taking no pressure outside the medium)

$$p(0) = 2\pi G \rho_0^2 d^2, \tag{5.6}$$

we may express the geometrical parameter d in terms of the physical parameters ρ_0 and $p(0)$, yielding

$$\lambda_c = \frac{1}{0.6392...\rho_0}\sqrt{\frac{2\pi p(0)}{G}}, \tag{5.7}$$

which is again λ_J up to a factor of about 2.

One may object that neither the pressure nor the field are homogeneous in the radial direction of the cylinder or in the direction perpendicular to the plane-parallel medium. However, this confirms rather what was stated in the preceding paragraph: small inhomogeneities do not matter.

5.3 Real Objects: Jeans's Length $\cong$ Diameter of Celestial Body

Another strong argument in favor of Jeans's criterion is by considering real objects. If an object (e.g. a star or a planet) is stable this means that its dimensions (say its diameter) are at most about λ_J, otherwise it might become unstable and break into smaller pieces. Callebaut (1967, 1986) applied this to many objects and found a fair agreement. E.g. for the Earth ($R_E = 6370$ km, $M_E = 6 \times 10^{24}$ kg) we have $\rho = 5.5 \times 10^3$ kg/m, $T \cong 3500$ K, $\lambda_J = 1.4 \times 10^7$ m and $M_J = 16 \times 10^{24}$ kg. Clearly, we have $\lambda_J \geq 2R_E$ and $M_J > M_E$, but the agreement is quite reasonable.

Similarly, for the Sun ($R_S = 7 \times 10^8$ m and $M_S = 2 \times 10^{30}$ kg) we have: (a) using values in the center: $\rho_c = 1.6 \times 10^5$ kg/m^3, $p_c = 10^{16}$ Pa, $T_c = 1.4 \times 10^7$ K, thus yielding $\lambda_J = 1.6 \times 10^8$ m and $M_J = 0.6 \times 10^{30}$ kg: reasonable values, but somewhat too small, as in fact the density is too high for the average, resulting in too small λ_J and M_J as explained at the end of section (2.5); (b) using average values: $\rho_{av} = 1.3 \times 10^3$ kg/m^3, $p_{av} = 10^{14}$ Pa, $T_{av} = 0.8 \times 10^7$ K, thus yielding $\lambda_J = 14 \times 10^8$ m and $M_J = 3.6 \times 10^{30}$ kg, values which are quite reasonable.

For galaxies we find reasonable results as well, but we have to take into account (a) the gas pressure; (b) the star pressure; (c) the radiation pressure (however, most of the radiation is leaking out of the galaxy, thus a reduced radiation pressure is in order); (d) the rotational pressure if it is a spiral galaxy. See Callebaut, 1986.

The observed tessellation of the universe is in a sense the inverse of the gravitational contraction: where matter has a density which is less than the average the opposite of a contraction occurs: it are the surroundings which contract and accumulate matter, while the rarefied region is even more rarefied and enlarged.

Let us take 1 hydrogen ion (+1 electron) per m^3, i.e. $\rho_U = 1.67 \times 10^{-27}$ kg/m^3 and $T_U = 2.7$ K for the cosmic temperature. There results

$$\lambda_{JU} = \frac{1}{\rho_0}\sqrt{\frac{2\pi\Gamma n_U k_B T_U}{2G}} \cong 8 \times 10^{20} m, \tag{5.8}$$

what is about 10^5 lightyear, about the diameter of our galactic disc.

On the other hand the radiation pressure is much greater than the usual kinetic pressure. We find

$$p_r = \frac{aT^4}{3} \cong 1,3 \times 10^{-14} J/m^3, \tag{5.9}$$

where $a = 7,56596 \times 10^{-16}$ J/m^{3}K^4 is the constant of Stefan-Boltzmann. Note that radiation pressure, although extremely low because of the low temperature, is about 3×10^8 higher than the kinetic pressure (which is abnormally low because of the extremely low density). This yields

$$\lambda'_{JU} = 1.3 \times 10^{25} m, \tag{5.10}$$

what is of the order of one tenth of the estimated dimensions of the universe. Of course, one should reconsider the derivation of Jeans's instability taking the radiation pressure into account.

5.4 Scale Adaptation for Uniform Variation

Jeans claimed that his instability criterion was at the base of a whole generation of celestial bodies: galaxies, star clusters, stars, planets and satellites. This is not quite correct as we will show now by an example; however, the spirit of his claim contains some truth as we explain afterwards. Let us consider a spherical symmetric star of homogeneous density ρ_0, radius R and mass $M = 4\pi\rho_0 R^3/3$. For the potential we have

$$\varphi(r) = \frac{2\pi G\rho_0 r^2}{3} - \frac{3GM}{2R} \qquad r \leq R, \tag{5.11}$$

$$\varphi^{ex}(r) = -\frac{GM}{r}, \qquad r \geq R. \tag{5.12}$$

The potential energy is

$$\Omega = -\frac{3GM^2}{5R}. \tag{5.13}$$

The pressure is

$$p(r) = 2\pi G\rho_0^2(R^2 - r^2)/3, \tag{5.14}$$

so that the pressure at the center is $p(0) = \frac{3GM^2}{8\pi R^4}$. Note that the average gravitational energy density in the star is about the same $\frac{3|\Omega|}{4\pi R^3} = \frac{9GM^2}{20\pi R^4}$ in absolute value. Using $p =$

nk_BT we find for the temperature distribution

$$T(r) = \frac{2\pi}{3k_B} G\mu H\rho_0(R^2 - r^2), \quad (5.15)$$

where μ is the molecular weight and H the mass of a hydrogen atom ($\rho_0 = n\mu H$). The radiation pressure has been neglected. The pressure and the temperature at the boundary are taken as zero. The Jeans's length for this sphere is according to Eq. (2.20)

$$\lambda_{Js} = \frac{1}{\rho_0}\sqrt{\frac{\pi\Gamma p_0}{2G}} = \frac{M}{4\rho_0 R^2}\sqrt{3\Gamma} = \pi R\sqrt{\frac{\Gamma}{3}}. \quad (5.16)$$

Here we have used the pressure at the center of the sphere. Eq. (5.16) confirms fully our hypothesis that the Jeans's length must be roughly the dimension of the body, i.e. about its diameter, as suggested in section 5.3.

Consider now a uniform transformation, e.g. one in which all lengths are multiplied by a factor q. This may be a contraction, as happens to interstellar clouds and protostars, or an expansion as may happen to stars because of the nuclear heat generation. If there are no heat losses nor heat generation we should put $\Gamma = \gamma = c_p/c_v$ where γ is the ratio of the specific heats. However, radiative losses are often important for gravitational instabilities and that leads to $\Gamma = 1$ or just a bit above 1 (as if the matter behaved isothermally), although radiation itself behaves in many aspects as if $\Gamma = 4/3$ (Chandrasekhar, 1939).

Applying a uniform variation we have $R' = qR$, $\rho' = \rho/q^3$, $M' = M$ (obviously invariant), $p' = p/q^4$, and $T' = T/q$. Cf. Lane's theorem (Chandrasekhar, 1939). Note that $p = K\rho^\Gamma$ would then require $\Gamma = 4/3$; however, we do not further investigate the consistency of the precise value of Γ. There results

$$\lambda'_{Js} = q\lambda_{Js}, \quad (5.17)$$

as follows upon substitution in Eq. (2.20) or directly from Eq. (5.16). This shows clearly that Jeans was wrong when he thought that generations of astronomical bodies were going to be generated by his criterion in a subsequent evolution: λ_J varies just like R, it is homothetically covariant with R, hence a contraction or an expansion will not generate new fractions. Nevertheless there is indirectly some truth in the idea of Jeans: in fact a body that contracts increases its temperature. However, if a large amount of the heat is radiated away, the pressure decreases accordingly and we get a λ'_J which becomes smaller than the new dimensions and then a new instability and smaller bodies may be generated. Actually this may rather start because the equilibrium conditions are no more satisfied and then the changes continue as an instability.

Our example is illustrative in an other aspect too: actually the pressure is lower near the boundary of the object. There fragmentation may occur, especially in shells, where the circumstances are similar (density, potential, pressure, temperature); see section (2.6) and the end of section (5.1). On the other hand in a real star the density is much lower near the boundary than in the center, so there is a mixture of counteracting factors active.

From the equation of dispersion it is clear that ω (or σ) varies like $q^{-3/2}$ and hence the characteristic time varies like $q^{3/2}$. Thus the characteristic times are shorter for more condensed configurations, and in fact faster than just linearly as one might expect from the smaller distances. In fact this is quite similar to the third law of Kepler which may be written as

$$\omega_p^2 D_p^3 = GM_t, \tag{5.18}$$

where ω_p is the angular frequency of the planet in its orbit around the central body, D_p its distance from the central body and M_t the total mass of the central body and planet. This makes clear that ω_p^2 (however, now corresponding to a rotation and not to a proper oscillation) is proportional to q^{-3}. The similarity with the dispersion equation (2.20) is even increased when introducing $\rho_{av} = 3M_p/4\pi D_p^3$ as the density averaged over the whole volume of the sphere (an approximation as the ellipticity is small) with radius D:

$$\omega_p^2 = \frac{4\pi G\rho_{av}}{3}. \tag{5.19}$$

Actually all this suggests another interpretation of Jeans's criterion. In some concrete cases (infinite cylinder, plane-parallel medium) Jeans's wavelength corresponds indeed to the critical transition between oscillations and instability. However, in most cases it corresponds to about the diameter of the astronomical body in equilibrium and it rather indicates that this body is stable: if the body were larger it would become unstable. However, due to a happy coincidence, the equilibrium does not allow that the body becomes larger without altering, at the same time and in the same sense, the instability length. The whole problem about Jeans's criterion is rather a question of equilibrium than one of stability.

5.5 Violation of Entropy Law on Cosmological Scale?

Jeans's instability contradicts the entropy law. Of course, one may argue that when a force occurs the entropy law may be violated. The point is that gravitation is innate to the matter, hence one can not pretend as if it does not exist. Jeans's instability seems to imply that the maximum of the entropy for a very large medium does not correspond to a homogeneous situation. This is precisely what is observed in astronomy: there are lumps of matter of all kinds: planets, stars, star clusters, galaxies, clusters of galaxies, the tessellation of the universe. All these contradict the homogeneous distribution of matter as desired by the entropy law. This may be seen as a warning when applying the entropy concept to astrophysical problems.

6 Conclusions

We have re-derived Jeans's criterion using the approximation of homogeneity around an extremum of the potential. However, we showed that the region of validity for this approximation is actually smaller than the Jeans's length, thus still not allowing a correct analysis. Using the customary cosmological term in a Newtonian approximation reproduced the Jeans's length asymptotically. Similarly, a rather complicated (and maybe unphysical) cosmological term did reproduce Jeans's criterion asymptotically as well. However, using another cosmological term (which we consider as more plausible) gave one fixed critical wavelength only, independent from the density. This does not correspond to Jeans's criterion. Hence, the idea vanishes that any small cosmological term which solves the inconsistency in the equilibrium will reproduce the Jeans's result to a good approximation. Moreover, other problems arise, e.g. how to reconcile the very minute average density of the universe with the density of a cloud which is going to generate a star? We recall that Lifshitz (1946), using general relativity, and Bonnor (1965), using Newtonian physics, did recover Jeans's wavelength in an expanding universe, however, the growth rates were much too small. We hit on a similar problem using various cosmological terms.

We gave three physical arguments which all three confirmed λ_J up to a (small) numerical factor. (a) When enough mass (hence a sufficiently large volume) is involved then gravitation ultimately wins over the pressure; (b) the infinite gravitational cylinder and the infinite gravitational plane-parallel medium have critical instability lengths which correspond to λ_J up to a factor of order unity; (c) real objects (e.g. Earth or Sun) have a diameter which corresponds to λ_J; moreover, we investigated a homothetic transformation. We have come to the conclusion that only in a few cases Jeans's criterion is an instability criterion. In most cases (considering actual bodies in equilibrium) it is rather the expression that the body in equilibrium is stable, λ_J being slightly larger than the diameter of the body, a happy feature of nature. No wonder that so many analyses trying to prove Jeans's instability failed.

Finally we wondered about the compatibility between the entropy law and gravitation (expressed by Jeans's wavelength and by the lumps in the universe).

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