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Adomian Decomposition Method Applied to Nonlinear Integral Equations

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Abstract
Our main objective in this paper is to study the accuracy, applicability and simplicity of Adomian method applied to non linear integral equations. In this work, we introduced a result obtained by the linearization method applied on a selected non-linear integral equation. Then, we compared this result against a result, we obtained by the Adomian decomposition method. This comparison study, showed the applicability and the accuracy of Adomian decomposition method comparing with the linearization method, even when the accuracy of linearization method improved by employing variable steps. This study showed also, that the simplicity and the speed of the convergent of Adomian decomposition method is depended on the initial choice of y_0.

1. Introduction

In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations such as implicitly linear collocation methods [1], product integration method [2], and Hermite-type collocation method [3] and semi analytical-numerical techniques such as Adomian decomposition method [4]. In this paper, we introduce a result obtained by the linearization method [5], then we compared this result against a result, we will obtain by Adomian method. To do that, we will introduce Adomian method for the nonlinear integral equations.

The general form of nonlinear integral equations which we will study is

$$y(x) = f(x) + \lambda \int_a^x k(x,t)\ H(y(t))dt \,. \tag{1.1}$$

In (1.1), $y(x)$ is an unknown function, a is a real constant. The kernel $k(x,t)$ and $f(x)$ are analytical functions on R^2 and R, respectively, λ is a real (or complex) parameter, known as the eigenvalue when λ is real parameter, and H is nonlinear function of y. Equation (1.1) represents a nonlinear Volterra integral equation of second kind.

The first step of Adomian technique is to decompose y into $\sum_{n=0}^{\infty} y_n$ and assume that

$$y = \lim_{n\to\infty} \left\{ \sum_{i=0}^{n} y_i \right\} \tag{1.2}$$

Next, we choose $y_0 = f(x)$ and set $H(y) = \sum_{n=0}^{\infty} A_n$, where A_n; $n \geq 0$ are special polynomials known as Adomian polynomials [6-7].

Now equation (1.1) becomes,

$$\sum_{n=0}^{\infty} y_n = f(x) + \lambda \int_a^x \left(k(x,t) \sum_{n=0}^{\infty} A_n \right) dt \,. \tag{1.3}$$

This leads to the recursive formulas

$$y_0 = f(x)$$

$$y_{n+1} = \lambda \int_a^x \left(k(x,t)\ A_n \right)\ dt \ , n = 0,\ 1,\ 2,\ \dots\ . \tag{1.4}$$

As we will see later, the choice of the initial data y_0, plays an essential rule on the speed of the convergence of Adomian method. In ref [9], close formulas of Adomian polynomials A_n for any analytic nonlinear function $H(y)$, introduced in the forms

$$A_0 = y(y_0),$$

$$A_n = \sum_{v=1}^{n} \left(\frac{1}{v!} \sum_{i_1, i_2, \dots, i_v = 1}^{n+1-v} \delta_{n, i_1 + i_2 + \dots + i_v} y_{i_1} y_{i_2} \dots y_{iv} \right) \frac{d^v H(y_0)}{dy_0^v}, \qquad n = 1, 2, \dots . \tag{1.5}$$

Here $n \ge v$ and $\delta_{n,m}$ is the kronecker delta. Consequently, this gives

$$A_0 = H(y_0)$$

$$A_1 = y_1 \frac{d}{dy_0} H(y_0)$$

$$A_2 = y_2 \frac{d}{dy_0} H(y_0) + \frac{1}{2\ !}\ y_1^2 \frac{d^2}{dy_0^2} H(y_0) \tag{1.6}$$

$$A_3 = y_3 \frac{d}{dy_0} H(y_0) + y_1 y_2 \frac{d^2}{dy_0^2} H(y_0) + \frac{1}{3\ !}\ y_1^3 \frac{d^3}{dy_0^3} H(y_0)$$

$$\vdots$$

To generate the highest order Adomian polynomials, we can use the formulas (1.5) and any Mathematical packages. This enables us to calculate any desire order of these polynomials. As example, we can found

$$A_7 = y_7 \frac{d}{dy_0} H(y_0) + (y_3 y_4 + y_2 y_5 + y_1 y_6)\ \frac{d^2}{dy_0^2} H(y_0) +$$

$$\left(\frac{y_2^2 y_3}{2} + \frac{y_1 y_3^2}{2} + y_1 y_2 y_4 + \frac{y_1^2 y_5}{2} \right) \frac{d^3}{dy_0^3}\ H(y_0) + \left(\frac{y_1 y_2^3}{6} + \frac{y_1^2 y_2 y_3}{2} + \frac{y_1^3 y_4}{6} \right) \frac{d^4}{dy_0^4}\ H(y_0) + \tag{1.7}$$

$$\left(\frac{y_1^3 y_2^2}{12} + \frac{y_1^4 y_3}{24} \right) \frac{d^5}{dy_0^5}\ H(y_0) + \frac{y_1^5 y_2}{120}\ \frac{d^6}{dy_0^6}\ H(y_0) + \frac{y_1^7}{5040}\ \frac{d^7}{dy_0^7}\ H(y_0).$$

Note that: the polynomial A_0 depends only on y_0, A_1 on y_0, y_1 and A_2 on y_0, y_1, y_2. In general the polynomials A_n depends on only $y_0, y_1, \dots, y_n$, with the sum of subscripts of the components of y_n of each term of A_n is equal to n. In addition, the polynomial A_0 is always determined independently of the other polynomials A_n ; ($n \geq 1$), hence A_0 is always defined by $A_0 = H(y_0)$.

2. Linearization-Method Result

The linearization method based on the piecewise linearization of the nonlinear integral equations, and the analytical solution of the resulting linear integral equation. In this section, we present a numerical result obtained by the linearization method. In ref. [8], they study the solution of the following Volterra non-linear integral equation in $[0,1]$

$$y(x) = e^x - \frac{1}{2}\left(e^{2x} - 1\right) + \int_0^x y^2(t)dt \tag{2.1}$$

Following the same reference, we can reduce the integral equation (2.1) to the linear integral equation

$$y(x) = e^x - \frac{1}{2}\left(e^{2x} - 1\right) - xy_n^2 + 2y_n \int_0^x y(t)dt, \qquad x_n \leq x < x_{n+1}\,. \tag{2.2}$$

The numerical solution of (2.2), with step size h and at the grid points $x_{n+1};\ (n = 0,1,2,.....)$, can be obtained from the formula

$$y_{n+1} = \frac{1}{2}y_n + e^{x_{n+1}} - \frac{1}{2}\left(e^{2x_{n+1}} - 1\right) + e^{2hy_n}\left(\frac{1}{2}y_n - e^{x_{n+1}} + \frac{1}{2}\left(e^{2x_{n+1}} - 1\right)\right) +$$

$$2y_n e^{2y_n x_{n+1}} \int_{x_n}^{x_{n+1}} \left(e^t - \frac{1}{2}\left(e^{2t} - 1\right)\right) e^{-2ty_n} dt \tag{2.3}$$

Following ref [7], Table 2.1, shows the errors-involved presented by the Linearization method with the step sizes 0.0001, 0.001, 0.01, and 0.1 along with the exact solution. The exact solution at x = 0.7 is 2.013752707000, whereas the numerical solutions corresponding to 0.0001, 0.001, 0.01, and 0.1 are 2.0137523240, 2.0137511060, 2.0136061360, and 2.0005244930, respectively. Note that: the aim of Darania Ebadian and Oskoi work [8], was to get $e^{x_r} \leq 10^{-k_r}$, where k_r is any positive integer. Hence, by assuming $Max\left(10^{-k_r}\right) = 10^{-k}$, the step size h can be decreasing as far as the inequality $e^{x_r} \leq 10^{-k}$ holds at each point x_r.

Table 2.1 shows the errors-involved presented by the Linearization method with h = 0.0001, 0.001, 0.01, and 0.1 along with the exact solution.

	h = 0.1	**h = 0.01**	**h = 0.001**	**h = 0.0001**
0.0	0.000000000 e +00	0.000000000 e +00	0.000000000 e +00	0.000000000 e +00
0.1	3.779750000 e -04	4.065000000 e -06	5.100000000 e -08	2.000000000 e -08
0.2	9.402320000 e -04	1.015300000 e -05	1.160000000 e -07	5.800000000 e -08
0.3	1.783404000 e -03	1.933700000 e -05	2.150000000 e -07	8.500000000 e -08
0.4	3.063988000 e -03	3.337100000 e -05	3.700000000 e -07	1.190000000 e -07
0.5	5.042530000 e -03	5.519100000 e -05	6.140000000 e -07	1.890000000 e -07
0.6	8.166182000 e -03	8.989100000 e -05	9.900000000 e -07	2.580000000 e -07
0.7	1.322821400 e -02	1.465710000 e -04	1.601000000 e -06	3.830000000 e -07
0.8	2.168746500 e -02	2.421720000 e -04	2.612000000 e -06	6.260000000 e -07
0.9	3.633225600 e -02	4.095920000 e -04	4.382000000 e -06	9.340000000 e -07
1.0	6.271307700 e -02	7.156940000 e -04	7.651000000 e -06	1.600000000 e -06

3. Adomian Decomposition Method Result

In order to assess both the applicability and accuracy of the Adomian method, we apply Adomian method on the selected non-linear integral equation (2.1), and compare our result against the linearization method result of the previous section. To do that, we follow ref. [7] and write (2.1) in the operator form

$$y(x) = e^{x} - \frac{1}{2}\left(e^{2x} - 1\right) + L_x\left(y^2(t)\right) \qquad L_t = \int_0^x (\cdot)\ dt\,, \tag{3.1}$$

For the Adomian polynomials A_n of the non-linear function $y^2(x)$ we can use the formula (1.5). This gives

$$\begin{aligned} A_0 &= y_0^2 \\ A_1 &= 2y_0y_1 \\ A_2 &= 2y_0y_2 + y_1^2 \\ A_3 &= 2y_0y_3 + 2y_1y_2 \\ A_4 &= 2y_0y_4 + 2y_1y_3 + y_2^2 \\ &\vdots \end{aligned} \tag{3.2}$$

Now, we decomposes y in (3.1) into $\sum_{n=0}^{\infty} y_n$, and equate the non-linear term to $\sum_{n=0}^{\infty} A_n$. Then (3.1) becomes

$$\sum_{n=0}^{\infty} y_n(x) = f(x) + L_x \sum_{n=0}^{\infty} A_n\,.$$

Finally, we set $y_0 = f(x)$, which yields the recursive formulas

$$\begin{aligned} y_0 &= f(x)\,, \\ y_n(x) &= L_x A_{n-1}\,, \qquad n \geq 1 \end{aligned} \tag{3.3}$$

This gives,

$$\begin{aligned} y_0 &= f(x) \\ y_1(x) &= L_x A_0 = L_x y_0^2 \\ y_2(x) &= L_x A_1 = L_x(2y_0y_1) \\ y_3(x) &= L_x A_2 = L_x(2y_0y_2 + y_1^2) \\ y_4(x) &= L_x A_3 = L_x(2y_0y_3 + 2y_1y_2) \end{aligned} \tag{3.4}$$

$$y_5(x) = L_x A_4 = L_x(2y_0y_4 + 2y_1y_3 + y_2^2)$$
$$\vdots$$

Now, (3.5) and (3.2) leads to

$$y_0(x) = \frac{1}{2} + e^x - \frac{1}{2}e^{2x}$$

$$y_1(x) = -\frac{47}{48} + \frac{1}{4}x + e^x + \frac{1}{4}e^{2x} - \frac{1}{3}e^{3x} + \frac{1}{16}e^{4x} \tag{3.5}$$

$$y_2(x) = \frac{551}{2880} - \frac{47}{48}x + \frac{1}{8}x^2 - \frac{35}{24}e^x + \frac{161}{96}e^{2x} - \frac{5}{18}e^{3x} - \frac{41}{192}e^{4x} + \frac{11}{120}e^5 - \frac{1}{96}e^{6x} + \frac{1}{2}xe^x - \frac{1}{8}xe^{2x}$$

Similarly, with the help of Mathematica Packages, we can calculate $y_n(x)$ for $n \geq 5$. This will lead to the approximate series solution

$$\tilde{y}_n(x) = y_0 + y_1 + y_3 + \ldots\ldots + y_n \tag{3.6}$$

Note that, we believe that the series in (3.6) converges [**] and

$$Lim_{n\to\infty}\tilde{y}_n(x) = \sum_{n=0}^{\infty} y_n(x) = e^x. \tag{3.7}$$

To compare our result against the linearization method result, we study the Adomian approximate solutions

$$\tilde{y}_n(x) = \sum_{i=0}^{n} y_i, \qquad n = 0,1,2,.... \tag{3.8}$$

For example:

$$\tilde{y}_1(x) = -\frac{23}{48} + \frac{1}{4}x + 2e^x - \frac{1}{4}e^{2x} - \frac{1}{3}e^{3x} + \frac{1}{16}e^{4x} \tag{3.9}$$

$$\tilde{y}_2(x) = -\frac{829}{2880} - \frac{35}{48}x + \frac{1}{8}x^2 + \frac{13}{24}e^x + \frac{137}{96}e^{2x} - \frac{11}{18}e^{3x} -$$
$$\frac{29}{192}e^{4x} + \frac{11}{120}e^5 - \frac{1}{96}e^{6x} + \frac{1}{2}xe^x - \frac{1}{8}xe^{2x} \tag{3.10}$$

The following table (3.1), contains Adomian approximate solutions $\tilde{y}_5(x), \tilde{y}_{10}(x)$, $\tilde{y}_{12}(x)$ and the exact solution $y(x) = e^x$ on the interval $[0,1]$. This result indicates that Adomian approximate solution $\tilde{y}_{12}(x)$ is nearly identical to the exact solution and we believe for large enough n, $\tilde{y}_n(x)$ is identical to the exact solution. It also showed that Adomian method minimizes the computational difficulties of the other methods, such as linearization method, because the components $y_0, y_1, y_2, \ldots$ are usually determined by using simple integration formulas.

Table 3.1: A comparison study between the numerical solutions of Adomian approximations $\tilde{y}_5(x)$, $\tilde{y}_{10}(x)$, $\tilde{y}_{12}(x)$ and the exact solution $y(x)$ on the interval $[0,1]$.

	Adomian approximation $\tilde{y}_5(x)$	....	**Adomian approximation** $\tilde{y}_{10}(x)$	....	**Adomian approximation** $\tilde{y}_{12}(x)$	**The Exact Solution** $y(x)=e^x$
0.0	1.000000000		1.000000000		1.000000000	1.000000000
0.1	1.105169816		1.105170918		1.105170918	1.105170918
0.2	1.221325536		1.221402734		1.221402757	1.221402758
0.3	1.348904080		1.349856590		1.349858612	1.349858808
0.4	1.486065716		1.491770554		1.491816326	1.491824698
0.5	1.625456162		1.648090633		1.648572331	1.648721271
0.6	1.749776851		1.817596906		1.820627063	1.822118800
0.7	1.827622422		1.990961309		2.003913672	2.013752707
0.8	1.812527615		2.138405580		2.178779637	2.225540928
0.9	1.649083627		2.194820246		2.290348556	2.459603111
1.0	1.288613226		2.058799960		2.234344769	2.718281828

4. The Convergence of Adomian Method

In this section study the convergence of Adomian method. In particular, we will show that the Adomian method has a fast convergent series solution comparing with other methods. We will also show that the speed of the convergent of Adomian decomposition method is depended on the initial choice of y_0. To do that, let us study integral equation (2.1) for different choice of y_0.

First, we write the integral equation (2.1) in the form

$$y(x) = 1 + \int_0^x \left(y^2(t) - e^{2t} + e^t \right) dt \tag{4.1}$$

Then we write (4.1) in the operator form

$$y(x) = 1 + L_t\left(y^2(t) - e^{2t} + e^t \right), \tag{4.2}$$

Next, we equate the non-linear term $y^2(x)$ to A_n, and decompose y into $\sum_{n=0}^{\infty} y_n$. Then (4.2) becomes

$$\sum_{n=0}^{\infty} y_n = 1 + L_t\left(\sum_{n=0}^{\infty}\left(A_n + \left(1-2^n\right)\frac{t^n}{n!}\right)\right)$$

For the initial data, we assume that $y_0 = 1$. This yields the recursive formulas

$$y_0 = 1,$$
$$y_{n+1} = L_t\left(A_n + \left(1-2^n\right)\frac{t^n}{n!}\right), \qquad n = 0,1,2,...., \tag{4.3}$$

This gives,

$$y_0 = 1$$
$$y_1(x) = L_t\left(A_0 + 0\right) = L_t\left(y_0^2\right) = L_t\left(1\right) = x$$
$$y_2(x) = L_t\left(A_1 - t\right) = L_t(2y_0y_1 - t) = L_t(2t - t) = \frac{1}{2!}x^2$$
$$y_3(x) = L_t\left(A_2 - \frac{3}{2}t^2\right) = L_t(2y_0y_2 + y_1^2 - \frac{3}{2}t^2) = L_t(t^2 + t^2 - \frac{3}{2}t^2) = L_t(\frac{1}{2}t^2) = \frac{1}{3!}x^3 \tag{4.4}$$
$$y_4(x) = L_t\left(A_3 - \frac{7}{3!}t^3\right) = L_t(2y_0y_3 + 2y_1y_2 - \frac{7}{3!}t^3) = L_t(\frac{2}{3!}t^3 + \frac{2}{2!}t^3 - \frac{7}{3!}t^3) = L_t(\frac{1}{3!}t^3) = \frac{1}{4!}t^4$$

$$y_5(x) = L_t\left(A_4 - \frac{15}{4!}t^4\right) = L_t(\frac{2}{4!}t^4 + \frac{2}{3!}t^4 + \frac{1}{2!\cdot 2!}t^4 - \frac{15}{4!}t^4) = L_t(\frac{2}{4!}t^4 + \frac{2}{3!}t^4 + \frac{1}{2!\cdot 2!}t^4 - \frac{15}{4!}t^4) == L_t(\frac{1}{4!}t^4) = \frac{1}{5!}t^5$$
$$\vdots$$

Next, we used the Mathematica Packages to calculate $y_n(x)$ for $n \geq 5$. This led to the approximate series solution

$$\tilde{y}_n\left(x\right) = y_0 + y_1 + y_3 + ... + y_n \tag{3.6}$$

This gives,

$$y(x) = Lim_{n->\infty}\tilde{y}_n\left(x\right) = \sum_{n=0}^{\infty}\frac{1}{n!}x^n = e^x$$

This result showed that the Adomian method has a fast convergent series solution comparing with other methods and the convergent of the series solution depends on the choice of initial data, y_0. In particular, this showed also the applicability and the accuracy of the Adomian method, comparing with the linearization method, even when a small number of iterations are used.

5. Conclusion

In this work we showed the accuracy, applicability and simplicity of Adomian method applied to non linear integral equations. A compression study against the linearization method, showed the applicability and the accuracy of Adomian decomposition method, even when the accuracy of linearization method improved by employing variable steps. This study also showed, that the speed of the convergent of Adomian series solution depends on the initial choice of y_0, which will open the door for further research in this direction. In particular, this study showed the accuracy of the Adomian method even when a small number of iterations are used. This encourages us to apply the same approach for other types of integral equations.

References

[1] H. Brunner, Implicitly linear collocation methods for nonlinear Volterra equations, Applied Numerical Mathematics **9** no. 3–5: 235–247 (1992).

[2] T. Tang, S. McKee, & T. Diogo, Product integration methods for an integral equation with logarithmic singular kernel, Applied Numerical Mathematics **9** no. 3–5: 259–266 (1992).

[3] T. Diogo, S. McKee, & T. Tang, A Hermite-type collocation method for the solution of an integral equation with a certain weakly singular kernel, IMA Journal of Numerical Analysis **11** no. 4: 595–605 (1991).

[4] A.-M. Wazwaz & S. M. El-Sayed, A new modification of the Adomian decomposition method for linear and nonlinear operators, Applied Mathematics and Computation, **122** no. 3: 393–405 (2001).

[6] G. Adomian, The Decomposition Method for Nonlinear Dynamical Systems, Journal of Mathematical Analysis and Applications, **120** no. 1: 370 – 383 (1986).

[7] G. Adomian, A Review of the Decomposition Method and Some Recent Results for Nonlinear Equations, Mathematical and Computer Modelling, **13** no. 7: 17 – 43 (1990).

[8] P. Darania, A. Ebadian, & A. Oskoi, **Linearization Method For Solving Non Linear Integral Equations**, Hindawi Publishing Corporation, Mathematical Problems in Engineering, Volume (2006), Article ID 73714, 1–10

[9] F. Abdelwahid, A Mathematical model of Adomian polynomials, Appl. Math. And Comp.**141:** 447-453 (2003).